BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VI Numéro 12

VARSOVIE 1958

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique.

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

Rédacteur en chef
K. KURATOWSKI

Rédacteur en chef suppléant
L. INFELD

SÉRIE DES SCIENCES MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Rédacteur de la Série
A. MOSTOWSKI

Comité de Rédaction de la Série
K. BORSUK, W. IWANOWSKA, A. JABŁOŃSKI, W. RUBINOWICZ, A. SOŁTAN

VOLUME VI
NUMÉRO 12

VARSOVIE 1958

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, WARSZAWA
MIODOWA 10

Nakład 846 + 124 egz.	*Rękopis dostarczono 4. XI. 1958*
Ark. wyd. 4,50 ark. druk. 4 1/2	*Podpisano do druku 17. I. 1958*
Papier druk. sat. 80 g kl. III	*Druk ukończono w styczniu 1959*
Format B5, 70×100 cm	*Zam. prod. 665/58 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHÉMATIQUE

Remarque sur un résultat de Kropatchev

par

D. PRZEWORSKA-ROLEWICZ

Présenté par K. BORSUK le 27 septembre 1958

Soit l'équation intégrale non-linéaire singulière:

$$u(t) = \lambda \int_L \frac{K[t, \tau, u(\tau)]\,d\tau}{\tau - t}, \tag{1}$$

où L désigne l'ensemble de p lignes fermées et disjointes sur le plan de la variable complexe, qui ont la tangente continue en tout point; $u(t)$ désigne une fonction inconnue. L'intégrale a le sens de la valeur principale de Cauchy.

Soit l'espace H^μ de toutes les fonctions qui vérifient la condition de Hölder avec un exposant fixé μ $(0 < \mu < 1)$:

$$|u(t) - u(t')| \leqslant c|t - t'|^\mu;$$

c est une constante positive arbitraire. L'espace H^μ est un espace de Banach, si nous admettons la définition suivante de la norme:

$$\|u\|_{H^\mu} = \sup_{t \in L} |u(t)| + \sup_{t, t' \in L} \frac{|u(t) - u(t')|}{|t - t'|^\mu}. \tag{2}$$

Soient les hypothèses suivantes:

(I) La fonction complexe $K(t, \tau, u)$ définie pour les variables complexes $t, \tau \in L$, $|u| \leqslant R$ vérifie la condition de Hölder suivante:

$$|K(t, \tau, u) - K(t', \tau', u')| \leqslant k[|t - t'|^\nu + |\tau - \tau'|^\mu + |u - u'|] \tag{3}$$
$$(0 < \mu < \nu \leqslant 1)$$

(II) Les fonctions

$$K^{(1)}(t, \tau, x_1, x_2) = \operatorname{Re} K(t, \tau, u)$$
$$K^{(2)}(t, \tau, x_1, x_2) = \operatorname{Im} K(t, \tau, u) \qquad (u = x_1 + ix_2)$$

admettent les dérivées partielles $K_{x_j}^{(n)}$ $(j, n = 1, 2)$, qui vérifient les conditions suivantes:

$$(4) \quad |K_{x_j}^{(n)}(t, \tau, x_1, x_2) - K_{x_j}^{(n)}(t', \tau', x_1', x_2')| \leqslant k_j^{(n)}[|t-t'|^{\nu} + |\tau-\tau'|^{\mu} + |x_1-x_1'| + |x_2-x_2'|] \quad (j, n = 1, 2),$$

$k_j^{(n)}$, k sont des constantes positives.

M. Kropatchev affirme dans son travail [1], que le résultat suivant est vrai:

"Si la fonction $K(t, \tau, u)$ vérifie l'hypothèse (I), alors l'équation (1) pour les valeurs suffisamment petites du paramètre λ admet la solution unique (dans la classe de fonctions vérifiant la condition de Hölder avec l'exposant μ), qui est une limite de la suite des approximations successives convergentes d'après la norme dans l'espace H^{μ}".

L'exemple suivant montre que ce n'est pas possible.

EXEMPLE. Soit la ligne L, qui se compose de l'intervalle fermé $[-1, 1]$ et de la ligne L', qui complète cette intervalle à la ligne fermée ayant la tangente continue en tout point.

Admettons

$$(4) \quad K(t, \tau, u) = \begin{cases} \operatorname{Re} u & \text{si} \quad \operatorname{Re} u \geqslant 0, \\ 0 & \quad\ \operatorname{Re} u < 0. \end{cases}$$

Définions les deux suites suivantes:

$$u_n(t) = \begin{cases} 1+t & \text{pour} \quad -1 \leqslant t \leqslant -1+\frac{1}{n}, \\ \frac{1}{n} & \quad\ -1+\frac{1}{n} \leqslant t \leqslant 1-\frac{1}{n}, \\ 1-t & \quad\ 1-\frac{1}{n} \leqslant t \leqslant 1, \\ 0 & \quad\ t \in L', \end{cases}$$

$$u_n^0(t) = u_n(t) - \frac{1}{n}.$$

Évidemment,

$$(5) \quad \|u_n(t) - u_n^0(t)\|_{H^{\mu}} = \frac{1}{n}.$$

Nous avons notamment

$$0 \leqslant u_n(t) \leqslant \frac{1}{n}, \quad \text{donc} \quad \operatorname{Re} u_n = u_n \geqslant 0; \quad K(u_n) = u_n$$

$$0 \geqslant u_n^0(t) \geqslant -\frac{1}{n}, \quad \text{donc} \quad \operatorname{Re} u_n^0 = u_n^0 < 0; \quad K(u_n^0) = 0,$$

d'où nous obtenons

$$\hat{K}u_n - \hat{K}u_n^0 = \int_L \frac{K[u_n(\tau)]}{\tau - t}\,d\tau - \int_L \frac{K[u_n^0(\tau)]}{\tau - t}\,d\tau = \int_L \frac{K[u_n(\tau)]}{\tau - t}\,d\tau .$$

Soit $t = -1$. Alors, d'après la définition de la suite $u_n(t)$:

$$\int_L \frac{u_n(\tau)}{\tau+1}\,d\tau = \int_{L'} \frac{u_n(\tau)}{\tau+1}\,d\tau + \int_{-1}^{-1+1/n} \frac{\tau+1}{\tau+1}\,d\tau + \frac{1}{n}\int_{-1+1/n}^{1-1/n} \frac{d\tau}{\tau+1} + \int_{1-1/n}^{1} \frac{1-\tau}{1+\tau}\,d\tau$$

$$\geqslant \frac{1}{n}\int_{-1+1/n}^{1-1/n} \frac{d\tau}{\tau+1} = \frac{1}{n}\left[\ln\left(2-\frac{1}{n}\right) - \ln\left(\frac{1}{n}\right)\right] \geqslant \frac{\ln n}{n}$$

si nous remarquons, que l'intégrale étendue sur L' est égale à zéro.

Il en résulte l'inégalité suivante (d'après (5)):

$$\|\hat{K}u_n - \hat{K}u_n^0\|_{H^\mu} \geqslant \frac{\ln n}{n} = \ln n\,\|u_n - u_n^0\|_{H^\mu} . \tag{6}$$

Alors la fonction $\hat{K}u = \int_L \frac{K[u(\tau)]}{\tau - t}\,d\tau$ (où $K[u(\tau)]$ est définie par les égalités (4)) ne vérifie pas la condition de Lipschitz d'après la norme dans l'espace H^μ.

J'ai démontré dans mon travail [2] par la méthode des approximations successives le théorème suivant:

THÉORÈME. *Si la fonction $K(t, \tau, u)$ vérifie les hypothèses* (I) *et* (II), *alors l'équation* (1) *pour les valeurs suffisamment petites du paramètre λ admet la solution unique, qui est une limite de la suite des approximations successives convergentes d'après la norme dans l'espace H^μ.*

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] W. F. Kropatchev, *Sur les équations intégrales non linéaires singulières* (en russe), Trudy Kazansk. Awiac. Inst. **30** (1954).

[2] D. Przeworska-Rolewicz, *Sur l'application de la méthode des approximations successives à l'équation intégrale à forte singularité*. Ann. Polon. Math. (sous presse).

Digitized by the Internet Archive
in 2025

https://archive.org/details/phytopathologia_1958_6_12

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHEMATICS

A General Scheme of the Notions of Independence in Mathematics

by

E. MARCZEWSKI

Presented on September 27, 1958

The purpose of this paper is to prove that many notions called *independence* in different branches of mathematics are particular cases of a common algebraic scheme. Let us note that this scheme is not new: the notion of independence defined below in No. 2, can be considered as a case of the notion of free $\mathfrak{A}$-algebra, defined by G. Birkhoff *). Nevertheless, the last notion will not be used explicitly in the sequel.

1. Algebras and algebraic operations. Let us consider an *algebra* **) $(A, \boldsymbol{F})$, i. e. a set A of *elements* together with a class $\boldsymbol{F}$ of *fundamental operations*. Each $f \in \boldsymbol{F}$ is a function of n variables (where n depends of f): it associates with every sequence $(x_1, \ldots, x_n)$ of elements of A an element $f(x_1, \ldots, x_n)$ of A. If $\boldsymbol{F} = \{f_1, \ldots, f_k\}$ we often write $(A; f_1, \ldots, f_k)$ instead of $(A, \boldsymbol{F})$.

We mean by the class of *algebraic operations* in $(A, \boldsymbol{F})$ the class $\boldsymbol{A}$ of all functions obtained by composition from the fundamental operations ***). More precisely, the class of algebraic operations of n variables is the smallest class $\boldsymbol{A}^{(n)}$ of functions satisfying the following conditions:

(a) If $h_1, \ldots, h_k \in \boldsymbol{A}^{(n)}$ and f is a fundamental operation of k variables, then the operation h defined by the formula

$$h(x_1, \ldots, x_n) = f\big(h_1(x_1, \ldots, x_n), \ldots, h_k(x_1, \ldots, x_n)\big) \quad \text{for} \quad x_j \in A$$

belongs to $\boldsymbol{A}^{(n)}$.

---

*) See Birkhoff [1], p. viii and McKinsey-Tarski [6], p. 170. Cf footnote *) on p. 733. Let us add that our scheme is essentially different from the notion of abstract linear independence introduced by Whitney [10].

) Cf. Birkhoff [1], p. vii. *) Cf. McKinsey-Tarski [6], p. 160.

(b) n "identity functions" of n variables:

$$e_k^{(n)}(x_1, \ldots, x_n) = x_k, \quad \text{where} \quad k = 1, 2, \ldots, n \quad \text{and} \quad x_j \in A,$$

belong to $\boldsymbol{A}^{(n)}$.

It is not difficult to prove that the class of all algebraic operations is closed with respect to composition of operations, introduction of new variables, identification of variables and their permutation:

(i) If $h_1, \ldots, h_k \in \boldsymbol{A}^{(n)}$, and $g \in \boldsymbol{A}^{(k)}$, then the operation h defined by the formula

$$h(x_1, \ldots, x_n) = g\big(h_1(x_1, \ldots, x_n), \ldots, h_k(x_1, \ldots, x_n)\big) \quad \text{for} \quad x_j \in A$$

belongs to $\boldsymbol{A}^{(n)}$.

(ii) If $g \in \boldsymbol{A}^{(n)}$ and

$$h(x_1, \ldots, x_n, x_{n+1}) = g(x_1, \ldots, x_n) \quad \text{for} \quad x_j \in A,$$

then $h \in \boldsymbol{A}^{(n+1)}$.

(iii) If $g \in \boldsymbol{A}^{(n+1)}$ and

$$h(x_1, \ldots, x_n) = g(x_1, \ldots, x_n, x_n) \quad \text{for} \quad x_j \in A,$$

then $h \in \boldsymbol{A}^{(n)}$.

(iv) If $g \in \boldsymbol{A}^{(n)}$, $(j_1, \ldots, j_n)$ is any permutation of $(1, \ldots, n)$ and

$$h(x_1, \ldots, x_n) = g(x_{j_1}, \ldots, x_{j_n}) \quad \text{for} \quad x_j \in A,$$

then $h \in \boldsymbol{A}^{(n)}$.

2. General notion of independence. Let $(A, \boldsymbol{F})$ be an algebra, $\boldsymbol{A}^{(n)}$ the corresponding class of algebraic operations of n variables, and $N \subset A$. We say that N is a set of *independent* elements, when for each sequence of different elements $a_1, \ldots, a_n \in N$ and for each pair $g, h \in \boldsymbol{A}^{(n)}$, if

$$g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n)$$

then g and h are identical in A.

The following equivalence follows from the properties (listed above) of algebraic operations:

(i) *N is a set of independent elements if and only if for any sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ of elements of N (not necessarily different), for any $g \in \boldsymbol{A}^{(m)}$ and $h \in \boldsymbol{A}^{(n)}$ and any mapping p of N into A, if*

$$g(a_1, \ldots, a_m) = h(b_1, \ldots, b_n)$$

then

$$g\big(p(a_1), \ldots, p(a_m)\big) = h\big(p(b_1), \ldots, p(b_n)\big).$$

Hence,

(ii) *N is a set of independent elements if and only if each mapping of N into A can be extended to a homomorphism of the subalgebra generated by N into A* *).

It is easy to see that

(iii) *If N is a set of independent elements, then no element a of N belongs to the subalgebra generated by the set $N \setminus \{a\}$.*

The converse implication is not true.

3. Linear and algebraic independence of numbers. In the algebra $(R^1; +)$ of all real numbers with addition as the unique fundamental operation, the class of all algebraic operations consists of all not vanishing identically linear forms with integral non negative coefficients. Thus, *the independence in $(R^1; +)$ coincides with the linear independence of real numbers.*

In the case of the algebra $(R^1; +, \cdot)$ of real numbers with two fundamental operations: addition and multiplication, the class of algebraic operations consists of all not vanishing identically polynomials without constant term and with non negative integral coefficients. Thus, *the independence in $(R^1; -, \cdot)$ coincides with the algebraic independence of real numbers* **).

4. Linear independence of vectors. Every vector space can be considered as an algebra $(V, \boldsymbol{F})$: each vector is then an element of the algebra, and the class $\boldsymbol{F}$ of fundamental operations consists of the addition of vectors and the multiplication of vectors by every scalar. (Thus, $\boldsymbol{F}$ contains one binary operation and infinitely many unitary operations). Obviously, *the independence in the algebra $(V, \boldsymbol{F})$ is equivalent to the linear independence in the considered vector space.*

5. Algebraic independence of polynomials. Let us denote by $\boldsymbol{P}$ the class of all real polynomials and by $\boldsymbol{P}^{(n)}$ the class of all real polynomials of n variables. We shall treat $\boldsymbol{P}^{(n)}$ as an algebra, defining the class $\boldsymbol{F}$ of fundamental operations as follows: addition, multiplication and all real constants (more precisely all operations of the form $Q(P_1, \ldots, P_k) = C$ which associate with every sequence $P_1, \ldots, P_k \in \boldsymbol{P}^{(n)}$, a constant polynomial $C \in \boldsymbol{P}^{(n)}$, i. e. a real function of the form $C(x_1, \ldots, x_n) = c$). The class of algebraic operations is thus in this case identical with the class $\boldsymbol{P}$ of all polynomials: every polynomial $P(x_1, \ldots, x_k)$ is here considered as

*) It is easy to state the relation between the notion of independence and that of a free $\mathfrak{A}$-algebra: N is a set of independent elements in (A, F) if and only if the subalgebra generated by N is a free $\mathfrak{A}$-algebra (with the set N of free generators), where $\mathfrak{A}$ consists of a single element (A, F). See e. g. Birkhoff [1], p. viii.

**) Different sets of algebraically independent real numbers were defined by Lebesgue, Steiniz and von Neumann (see [7]).

a function which associates with every sequence $P_1, \ldots, P_k \in \boldsymbol{P}^{(n)}$ the polynomial

$$P\big(P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n)\big)$$

belonging also to $\boldsymbol{P}^{(n)}$.

Clearly, *the independence in the algebra* $(\boldsymbol{P}^{(n)}, \boldsymbol{F})$ *is equivalent to the independence of polynomials in the classical algebraic sense* *).

6. Logical independence. The well-known notion of independence of axioms (as well as the independence of equations, the independence of conditions, etc.) can be formulated in terms of the General Theory of Sets. Let us denote by 2^X the class of all subsets of a fixed set X, and let us suppose that $\boldsymbol{N} \subset 2^X$. We say that $\boldsymbol{N}$ is a class of *implicationally independent sets* if no intersection $E_1 \cap \ldots \cap E_n$ of sets belonging to $\boldsymbol{N}$ is contained in another set $E_{n+1} \in \boldsymbol{N}$. It is easy to see that *the independence in the algebra* $(2^X; \cap)$ *is equivalent to the implicational independence.*

There exists a notion dual to the implicational independence: no union $E_1 \cup \ldots \cup E_n$ contains another set $E_{n+1} \in \boldsymbol{N}$ (see [5], p. 86). Obviously, this notion is equivalent to *the independence in the algebra* $(2^X; \cup)$.

7. Set-theoretical independence. Let us consider the class 2^X of all subsets of a fixed set X (or, more generally, a Boolean algebra). We denote by $'$ complementation with respect to X, and we put $S^0 = S'$ and $S^1 = S$ for each set $S \subset X$. Each set of the form

$$(*) \qquad \bigcap_{k=1}^{n} E_k^{i_k}, \qquad \text{where} \qquad i_k = 0 \text{ or } 1$$

is called an atom of the family $E_1, \ldots, E_n$.

Let us recall that a class $\boldsymbol{N} \subset 2^X$ is called a class of independent sets (in the sense of the General Theory of Sets) whenever each atom of each finite subclass of $\boldsymbol{N}$ is non void **). It is not difficult to prove that *the independence in this sense is equivalent to the independence in the algebra* $(2^X; \cup, ')$.

There exist some interesting variants of the notion in question. In the case of an infinite class $\boldsymbol{N}$ of sets we may replace complementa-

*) For the algebraic independence of polynomials cf. e. g. [8], p. 124-140. S. Świerczkowski proved recently that the classical notion of independence of differentiable functions of several variables can also be treated as a case of our scheme. For other results of S. Świerczkowski concerning the notion of independence see [9].

**) The notion introduced by Fichtenholz and Kantorovitch ([2], p. 78). Cf. also [3] and [4] and the papers quoted there. Obviously, the so-called denumerable independence of sets (cf. e. g. [4], p. 123) could also be treated as the independence in an algebra whenever we consider some functions of infinitely many variables as fundamental operations.

tion in the preceding proposition by subtraction or intersection. In the case of a finite class, however, the three corresponding notions of independence are not equivalent. We have namely the following three propositions:

(i) *The sets* $E_1, \ldots, E_n$ *are independent in the algebra* $\mathfrak{B} = (2^X; \cup, ')$ *if and only if each atom* $(*)$ *is non-void.*

(ii) *The sets* $E_1, \ldots, E_n$ *are independent in the algebra* $\mathfrak{B}_1 = (2^X; \cup, \ )$ *if and only if each atom* $(*)$ *with not all vanishing upper indices* i_j *is non-void.*

(iii) *The sets* $E_1, \ldots, E_n$ *are independent in the algebra* $\mathfrak{B}_2 = (2^X; \cup, \cap)$ *if and only if each atom* $(*)$ *with not all identical upper indices* i_j *is non-void.*

Let us put further $\mathfrak{B}_3 = (2^X; \cup)$ and $\mathfrak{B}_4 = (2^X; \cap)$. It is easy to see that:

(iv) *In a set* X *there exist* n *subsets independent in the algebra*

$$\mathfrak{B}, \quad \vec{\mathfrak{B}}_1, \quad \vec{\mathfrak{B}}_2, \quad \mathfrak{B}_3, \mathfrak{B}_4$$

if and only if X *contains at least*

$$2^n, 2^n - 1, 2^n - 2, n, n$$

elements, respectively.

8. Independence of Boolean polynomials. For an algebra $(A, \boldsymbol{F})$ let us denote by $\boldsymbol{A}^{(n)}$, as before, the class of all algebraic operations of n variables. Since the values of all algebraic operations in $(A, \boldsymbol{F})$ belong to A, we may treat every operation belonging to $\boldsymbol{F}$ as an operation defined for variables running over $\boldsymbol{A}^{(n)}$. Thus, we obtain the algebra $(\boldsymbol{A}^{(n)}, \boldsymbol{F})$ of algebraic functions in $(A, \boldsymbol{F})$ (cf. [6], p. 160). E. g. if $(R^1, \boldsymbol{F})$ is the algebra of all real numbers with addition, multiplication and all real constants — as fundamental operations — then the corresponding algebra of algebraic operations of n variables is identical with the algebra $(\boldsymbol{P}^{(n)}, \boldsymbol{F})$ discussed above under No. 5.

In the case of the algebra $(2^X; \cup, ')$ the algebraic operations are called usually Boolean polynomials. It is well known that every set S of n-tuples consisting of 0's and 1's determines a Boolean polynomial:

$$(**) \qquad g(E_1, \ldots, E_n) = \bigcup_{(i_1, \ldots, i_n) \in S} \bigcap_{k=1}^{n} E_k^{i_k}$$

and that, conversely every Boolean polynomial has a unique canonical representation $(**)$.

There exists the following relation between the independence of sets and the independence of Boolean polynomials:

Let $g_1, \ldots, g_n$ *be Boolean polynomials* $(**)$ *of* n *variables determined by the sets* $S_1, \ldots, S_m$ *respectively. The operations* $g_1, \ldots, g_m$ *are independent if and only if the sets* $S_1, \ldots, S_m$ *are independent in the set-theoretical sense.*

9. Stochastic independence. The stochastic independence (i. e. the independence in the sense of probability) has a different character from all notions considered above *). The definition of the stochastic independence of the sets E_1 and E_2:

$$\mu(E_1 \cap E_2) = \mu(E_1) \cdot \mu(E_2)$$

requires the equality of two non identical real functions of two set-variables.

Thus, it seems to be quite natural that *the stochastic independence does not fall under the scheme defined in No* 2:

If μ is a probability-measure in a field $\boldsymbol{M}$ of sets, if $(\boldsymbol{M}, \boldsymbol{F})$ is an algebra with addition of sets belonging to $\boldsymbol{F}$, then the stochastic independence (with respect to μ) does not coincide with the independence in $(\boldsymbol{M}, \boldsymbol{F})$.

In fact, the empty set and any non-empty set form a pair of sets stochastically independent, but non-independent in the algebra $(\boldsymbol{M}, \boldsymbol{F})$.

INSTITUTE OF MATHEMATICS (WROCŁAW BRANCH), POLISH ACADEMY OF SCIENCES

REFERENCES

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Public., New York, **25** (1948).

[2] G. Fichtenholz et L. Kantorovitch, *Sur les opérations dans l'espace des fonctions bornées*, Studia Mathem. **5** (1934), 69-98.

[3] E. Marczewski, *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fund. Mathem. **35** (1948), 13-28.

[4] — *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, Colloq. Mathem. **1** (1948), 122-132.

[5] S. Mazurkiewicz, *Podstawy rachunku prawdopodobieństwa*, Monografie Matematyczne, Warszawa, **32** (1956).

[6] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Annals of Math., **45** (1944), 141-191.

[7] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann., **99** (1928), 134-141.

[8] O. Perron, *Algebra. I. Die Grundlagen*, Göschens Lehrbücherei, Berlin and Leipzig, 1932.

[9] S. Świerczkowski, *On independent elements in finitely generated algebras* (this issue).

[10] H. Whitney, *The abstract properties of linear dependence*, Am. Journal of Math., **57** (1935), 509-533.

*) In spite of its known connections with the set-theoretical independence (see [3] and [4]).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHÉMATIQUE

Sur les notions de distribution et d'opérateur

par

J. MIKUSIŃSKI

Présenté par K. BORSUK le 30 septembre 1958

Les distributions ont été définies par S. L. Soboleff [6] et L. Schwartz [5] comme fonctionnelles linéaires définies sur certains espaces de fonctions. Cependant, les opérateurs au sens de J. Mikusiński [1] sont des fractions, obtenues par extension d'un anneau de fonctions (avec convolution comme produit). Donc, du point de vue logique, les deux notions sont tout-à-fait différentes. Il arrive toutefois de dire qu'elles sont mathématiquement très proches et, même, que certaines distributions sont des opérateurs et réciproquement. D'autre part, on dit que certaines distributions ne sont pas des opérateurs et que certains opérateurs ne sont pas des distributions. Quel sens mathématique peut être attribué à de telles énonciations? — nous allons le préciser dans cette note.

Toute distribution est une limite, au sens distributionnel, d'une suite de fonctions continues (voir [4]). Pareillement, tout opérateur est une limite, au sens opérationnel, d'une suite de fonctions continues. Nous dirons qu'une distribution est égale à un opérateur lorsqu'il existe une suite de fonctions continues qui converge distributionnellement vers la distribution et opérationnellement vers l'opérateur. Cette définition est justifiée par le fait que l'espace des fonctions continues est dense dans l'espace des distributions et dense dans l'espace des opérateurs.

Les fonctions sont ici considérées dans l'intervalle $(-\infty, \infty)$. Si une suite de telles fonctions converge opérationnellement, chacune de ces fonctions est elle-même un opérateur, donc une fonction nulle dans un certain intervalle $(-\infty, t_0)$. Le cas $t_0 < 0$ n'est pas exclu, car toute fonction f, nulle dans $(-\infty, t_0)$, peut s'écrire $f = e^{-st_0} f_0$, où e^{-st_0} est l'opérateur de translation et f_0 une fonction nulle dans $(-\infty, 0)$.

Par exemple la distribution delta de Dirac est égale à l'opérateur unité, en symbole

$$\{\delta(t)\} = 1;$$

en effet, la suite des fonctions f_n, représentées par le diagramme ci-joint, converge distributionnellement vers δ et opérationnellement vers 1. Pareillement on a

$$\{\delta'(t)\} = s\,,$$

où s est l'opérateur différentiel.

Nous nous appuierons dans la suite sur les critères suivants de convergence:

(I) *Une suite de fonctions $f_n(t)$ continues dans l'intervalle $(-\infty, \infty)$ converge distributionnellement, lorsqu'il existe pour tout intervalle borné $[a, b]$ un entier $k \geqslant 0$ et une suite de fonctions $F_n(t)$, convergeant uniformément sur $[a, b]$, telle que $\frac{\partial^k}{\partial t^k} F_n(t) = f_n(t)$ sur $[a, b]$.*

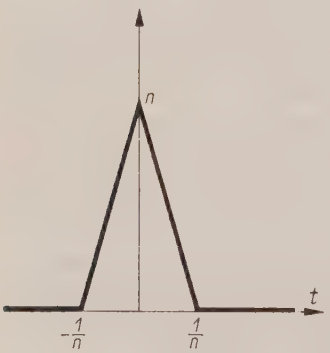

Fig. 1

(II) *Une suite de fonctions $f_n(t)$ continues dans l'intervalle $(-\infty, \infty)$ converge opérationnellement, lorsque chacune de ces fonctions est nulle dans un intervalle commun $(-\infty, t_0)$ et qu'il existe une fonction $f_0(t)$ continue dans $(-\infty, \infty)$, et non-identiquement nulle dans $(0, \infty)$, telle que la suite des produits de compositions*

$$(1) \qquad \int_{-\infty}^{\infty} f_n(t-\tau) f_0(\tau)\, d\tau \qquad (n = 1, 2, \ldots)$$

converge uniformément sur tout intervalle borné $[a, b]$.

Les conditions précédentes sont nécessaires et suffisantes. La démonstration est facile et ne sera pas reproduite ici.

Nous démontrerons cependant la proposition suivante:

(III) *Pour qu'une distribution soit un opérateur, il faut et il suffit que son support soit borné inférieurement.*

Démonstration. Une distribution a son support borné inférieurement, lorsqu'il existe une suite de fonctions continues $f_n(t)$, convergeant distributionnellement vers cette distribution, telle que $f_n(t) = 0$ dans un intervalle commun $(-\infty, t_0)$. Cette condition est nécessaire et suffisante. En prenant pour $f_0(t)$ une fonction indéfiniment dérivable dans $(-\infty, \infty)$, nulle dans $(-\infty, 0)$ et non-identiquement nulle dans $(0, \infty)$, il est facile de démontrer que la convergence distributionnelle de $f_n(t)$ entraîne la convergence de (1). Donc, une distribution à support borné inférieurement est un opérateur. D'autre part, si une distribution est un opérateur, il existe une suite f_n satisfaisant aux deux conditions (I) et (II), ce qui entraîne que la distribution est bornée inférieurement.

D'après la condition précédente, les distributions dont le support n'est pas borné inférieurement ne sont pas des opérateurs. La fonction constante à valeur 1 dans $(-\infty, \infty)$ est un exemple d'une telle distribution; cette fonction n'est donc pas un opérateur.

La condition (III) est très commode et permet de décider tout de suite, si une distribution est un opérateur ou non. La question de donner une condition inverse est plus difficile. Nous ne connaissons pas de critère pratique qui permettrait de décider, si un opérateur donné dans la forme a/b est une distribution. Même, il n'est pas trop trivial de donner un exemple d'un opérateur qui n'est pas une distribution.

On peut se douter que par exemple l'opérateur $e^{\sqrt{s}} = 1/f$, où

$$f(t) = \begin{cases} 0 & \text{pour} \quad t \leqslant 0 \\ \dfrac{1}{2\sqrt{\pi t^3}} \exp\left(\dfrac{-1}{4t}\right) & \end{cases},$$

est un tel opérateur (voir [2] et [3]). Nous allons le démontrer rigoureusement.

Nous nous appuierons sur deux lemmes:

(IV) *Le produit (au sens opérationnel) de deux opérateurs qui sont des distributions est encore une distribution, égale à la convolution des deux distributions.*

Par exemple:

$$1. = 1\left\{\int_{-\infty}^{\infty} \delta(t-\tau)\delta(\tau)\,d\tau\right\} = \{\delta(t)\}.$$

(V) *Si l'opérateur a est une distribution, le produit sa (s — opérateur différentiel) est encore une distribution, égale à la dérivée de cette distribution.*

Par exemple:

$$s \cdot 1 = \{\delta'(t)\}.$$

La démonstration de ces lemmes est très facile et se réduit à leur vérification, conformément à la définition adoptée de l'égalité entre un opérateur et une distribution.

Considérons maintenant la fonction exponentielle

$$e^{-\sqrt{s}\lambda}, \tag{2}$$

en admettant les valeurs complexes de λ. On a

$$e^{-\sqrt{s}\lambda} = \left\{ \begin{array}{ll} 0 & \text{pour} \quad t \leqslant 0 \\ \dfrac{\lambda}{2\sqrt{\pi t^3}} \exp\left(-\dfrac{\lambda^2}{4t}\right) & \text{pour} \quad t > 0 \end{array} \right\} \quad \text{dans l'angle} \quad |\arg\lambda| < \frac{\pi}{4}, \tag{3}$$

$$\frac{e^{-\sqrt{s}\lambda}}{\sqrt{s}} = \left\{ \begin{array}{ll} 0 & \text{pour} \quad t \leqslant 0 \\ \dfrac{1}{\sqrt{\pi t}} \exp\left(-\dfrac{\lambda^2}{4t}\right) & \text{pour} \quad t > 0 \end{array} \right\} \quad \text{dans l'angle} \quad |\arg\lambda| < \frac{\pi}{4}. \tag{4}$$

L'opérateur

$$\frac{e^{-\sqrt{s}\lambda}}{s} \tag{5}$$

peut être considéré comme la convolution de la fonction (4) et de la fonction

$$\frac{1}{\sqrt{s}} = \begin{cases} 0 & \text{pour} \quad t \leqslant 0 \\ \dfrac{1}{\sqrt{\pi t}} & \text{pour} \quad t > 0 \end{cases},$$

il est donc une fonction intégrable, discontinue au point $t = 0$ et nulle pour $t < 0$.

L'opérateur $e^{-\sqrt{s}\lambda}$ est le produit de (5) et de l'opérateur différentiel s, donc, ses valeurs sur les demi-droites

$$|\arg\lambda| = \frac{\pi}{2}$$

sont des distributions. Ces valeurs sont des dérivées des fonctions discontinues, elles ne sont donc pas elles-mêmes des fonctions.

Nous démontrerons que *les valeurs de la fonction* (2) *sont, pour* $\frac{\pi}{4} < |\arg\lambda| \leqslant \pi$, *des opérateurs qui ne sont pas des distributions.*

En effet, supposons que, pour un $\lambda_0 \neq 0$ tel que $\frac{\pi}{4} < |\arg\lambda_0| \leqslant \pi$, l'opérateur

$$e^{-\sqrt{s}\lambda_0} \tag{6}$$

soit une distribution. Soit $\lambda_0 = \lambda_1 + i\lambda_2$ (λ_1, λ_2 réels). Alors $\lambda_2 - \lambda_1 > 0$ et l'opérateur

$$e^{-\sqrt{s}(|\lambda_2| - \lambda_1)}$$

est, d'après (3), une fonction indéfiniment dérivable dans $-\infty < t < \infty$. La convolution de cette fonction avec la distribution (6) est, par conséquent, encore une fonction indéfiniment dérivable dans $-\infty < t < \infty$. D'autre part, cette convolution est égale à

$$e^{-\sqrt{s}(|\lambda_2| + i\lambda_2)}.$$

Comme $|\arg(|\lambda_2| + i\lambda_2)| = \frac{\pi}{4}$, il s'ensuit que cette convolution est une distribution qui n'est pas une fonction. Cette contradiction prouve que (6) ne peut pas être une distribution, ce qui achève la démonstration.

Plus généralement, on a la proposition suivante:

La valeur de $e^{-s^\alpha\lambda}$ $(0 < \alpha < 1)$ *est*

1° *une fonction indéfiniment dérivable, dans l'angle* $|\arg\lambda| < \frac{\pi(1-\alpha)}{2}$ $(|\lambda| > 0)$;

2° *une distribution qui n'est pas une fonction, sur les demi-droites* $\arg\lambda = \dfrac{\pi(1-\alpha)}{2}$;

3° *un opérateur qui n'est pas une distribution ni la fonction dans le domaine restant* $\dfrac{\pi(1-\alpha)}{2} < |\arg\lambda| \leqslant \pi (|\lambda| > 0)$.

La démonstration est analogue à celle du cas particulier $\alpha = \frac{1}{2}$, considéré tout à l'heure.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] J. Mikusiński, *Sur les fondements du calcul opératoire*, Studia Math. **11** (1950), 41-47.

[2] — *Sur les fonctions exponentielles du calcul opératoire*, Studia Math. **12** (1951), 208-224.

[3] — *On the function whose Laplace-transformation is* $e^{-s^{\alpha}}$, Studia Math. (à paraître).

[4] J. Mikusiński et R. Sikorski, *The elementary theory of distributions*, Rozprawy Matematyczne **12** (1957).

[5] L. Schwartz, *Théorie des distributions* t. I, Paris, 1950.

[6] S. L. Soboleff, *Méthode nouvelle a résoudre le problème de Cauchy pour les équations hyperboliques normales*, Recueil Math., **1** (1936), 39-71.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHEMATICS

On Sets Which Are Measured by Multiples of Irrational Numbers

by

P. ERDÖS and K. URBANIK

Presented by E. MARCZEWSKI on October 10, 1958

The frequency of naturals n satisfying a condition Φ is defined as the limit

$$\operatorname{fr}\{n : n \text{ satisfies } \Phi\} = \lim_{N\to\infty} \frac{1}{N} \overline{\overline{\{n : n \text{ satisfies } \Phi,\ n \leqslant N\}}}$$

provided this limit exists. ($\overline{\overline{A}}$ denotes the power of A).

We say that a set A ($A \subset [0, 1)$) belongs to *the class* Ξ if for every irrational ξ the frequency $\operatorname{fr}\{n : n\xi \in A (\mathrm{mod}\, 1)\}$ exists and does not depend on the choice of ξ. It is well-known that every Jordan measurable set belongs to Ξ and, moreover, the frequencies $\operatorname{fr}\{n : n\xi \in A (\mathrm{mod}\, 1)\}$ are equal to the measure of A. Further, it is easy to verify that every Hamel base (mod 1) belongs to Ξ, which shows that sets belonging to Ξ may be Lebesgue non-measurable.

We say that a class Ξ_0 is *the base* of the family Ξ if for every $A \in \Xi$ there exists a set $B \in \Xi_0$ such that

$$\operatorname{fr}\{n : n\xi \in A \dot{-} B (\mathrm{mod}\, 1)\} = 0 \text{ *)}$$

for any irrational ξ.

We say that a class Ξ_1 is *the weak base* of the family Ξ if for every $A \in \Xi$ there exists a set $B \in \Xi_1$ such that

$$\operatorname{fr}\{n : n\xi \in A \dot{-} B \ (\mathrm{mod}\, 1)\} = 0$$

for almost all ξ.

The purpose of this note is the investigation of Lebesgue measurability of sets belonging to a base or to a weak base of the family Ξ. Namely, we shall prove with the aid of the axiom of choice

*) $A \dot{-} B$ denotes the symmetric difference of the sets A and B.

THEOREM 1. *Every base of the family Ξ contains $2^{2^{\aleph_0}}$ Lebesgue non-measurable sets.*

THEOREM 2. *Every weak base of the family Ξ contains at least $2^{\aleph_1}$ Lebesgue non-measurable sets.*

COROLLARY. *Under assumption of the continuum hypothesis every weak base of the family Ξ contains $2^{2^{\aleph_0}}$ Lebesgue non-measurable sets.*

Before proving the Theorems, we shall prove three Lemmas. Let us introduce the following notations

$$\mathcal{U}_k = \{x : x \text{ rational, } k(k-1) \leqslant x < k^2\} \quad (k = 1, 2, \ldots),$$

$$\mathcal{W}_+ = \bigcup_{k=1}^{\infty} \mathcal{U}_k, \quad \mathcal{W}_- = \{x : -x \in \mathcal{W}_+\}, \quad \mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_-. \tag{1}$$

LEMMA 1. *For every rational number $r \neq 0$ the equality*

$$\operatorname{fr}\{n : nr \in \mathcal{W}\} = \tfrac{1}{2}$$

is true.

Proof. It is sufficient to prove that, for every positive rational number r, the equality $\operatorname{fr}\{n : nr \in \mathcal{W}_+\} = \frac{1}{2}$ holds.

Let $I^{(k)}(r)$ denote the number of such naturals n that $nr \in \mathcal{U}_k$.

Obviously,

$$\left[\frac{k}{r}\right] - 1 \leqslant I^{(k)}(r) \leqslant \left[\frac{k}{r}\right] + 1, \tag{2}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

$I_N(r)$ will denote the number of such naturals n ($n \leqslant N$) that $nr \in \mathcal{W}_+$. If $k \leqslant \lfloor \sqrt{Nr} \rfloor$ and $nr \in \mathcal{U}_k$ then, in view of (1), $nr < k^2 \leqslant \lfloor \sqrt{Nr} \rfloor^2 \leqslant Nr$, which implies the inequality $n < N$. Hence, we obtain the inequality

$$I_N(r) \geqslant \sum_{k=1}^{[\sqrt{Nr}]} I^{(k)}(r) \quad (N = 1, 2, \ldots).$$

Consequently, taking into account (2), we have the inequality

$$I_N(r) \geqslant \sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] - \lfloor \sqrt{Nr} \rfloor \quad (N = 1, 2, \ldots). \tag{3}$$

Further, if $k > \lfloor \sqrt{Nr} \rfloor + 1$ and $nr \in \mathcal{U}_k$ then, in view of (1), $nr \geqslant k(k-1) > Nr$, which implies the inequality $n > N$. Hence, we obtain the following inequality

$$I_N(r) \leqslant \sum_{k=1}^{[\sqrt{Nr}]+1} I^{(k)}(r) \quad (N = 1, 2, \ldots).$$

Consequently, taking into account (2), we have the inequality

$$I_N(r) \leqslant \sum_{k=1}^{[\sqrt{Nr}]+1} \left[\frac{k}{r}\right] + [\sqrt{Nr}] + 1 \quad (N = 1, 2, \ldots).$$

Hence, and from (3), it follows that

$$I_N(r) = \sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] + o(N) \quad (N = 1, 2, \ldots). \tag{4}$$

Setting $r = \frac{p}{q}$, $[\sqrt{Nr}] = d_N p + s_N$ $(0 \leqslant s_N < p)$, where p, q, d_N and s_N are integers we obtain by simple reasoning

$$\sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] = \frac{1}{2} pq d_N (d_N - 1) + d_N \sum_{j=1}^{p} \left[\frac{j}{r}\right] + \sum_{j=1}^{s_N} \left[\frac{j}{r}\right] + q d_N s_N$$

$$= \frac{1}{2} pq d_N^2 + o(N) = \frac{1}{2} N + o(N).$$

Hence, in virtue of (4), we obtain the equality $I_N(r) = \frac{1}{2} N + o(N)$. The Lemma is thus proved.

By γ we denote the first ordinal number of the power continuum. Let us consider a Hamel base $x_0 = 1, x_1, x_2, \ldots, x_\alpha, \ldots$ $(\alpha < \gamma)$. Every irrational number x may be represented as a linear combination with rational coefficients $x = r_0 + r_1 x_{\alpha_1} + \ldots + r_n x_{\alpha_n}$, where $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n$, $r_1 \neq 0$. In the sequel we shall use the notations $r(x) = r_1$, $\alpha(x) = \alpha_1$.

Let $\mathfrak{B}$ be the class of all subsets of the set of all positive ordinals less than γ. Obviously,

$$\overline{\overline{\mathfrak{B}}} = 2^{2^{\aleph_0}}. \tag{5}$$

For every $V \in \mathfrak{B}$ we define the set

$$A_V = \{x : x \text{ irrational}, \ 0 < x < 1, \ r(x) \in \mathcal{W}, \ \alpha(x) \in V\} \cup$$
$$\{x : x \text{ irrational}, \ 0 < x < 1, \ r(x) \text{non} \in \mathcal{W}, \ \alpha(x) \text{non} \in V\}.$$

LEMMA 2. *For every $V \in \mathfrak{B}$ $A_V \in \Xi$. Moreover,*

$$\operatorname{fr}\{n : n\xi \in A_V (\operatorname{mod} 1)\} = \tfrac{1}{2}$$

for each irrational ξ.

Proof. Since $r(n\xi) = nr(\xi)$ and $\alpha(n\xi) = \alpha(\xi)$ we have the following equality

$$\{n : n\xi \in A_V (\operatorname{mod} 1), \ n \leqslant N\} = \begin{cases} \{n : nr(\xi) \in \mathcal{W}, \ n \leqslant N\} & \text{if} \quad \alpha(\xi) \in V, \\ \{n : nr(\xi) \text{non} \in \mathcal{W}, \ n \leqslant N\} & \text{if} \quad \alpha(\xi) \text{non} \in V. \end{cases}$$

Hence, according to Lemma 1, for every irrational ξ, we obtain the equality $\operatorname{fr}\{n: n\xi \in A_V (\mathrm{mod}\, 1)\} = \frac{1}{2}$, which was to be proved.

LEMMA 3. *Let D $(D \subset [0,1))$ be a set satisfying the equality*

$$\operatorname{fr}\{n: n\xi \in A_V \dot{-} D (\mathrm{mod}\, 1)\} = 0 \qquad (V \in \mathfrak{B}) \tag{6}$$

for almost all ξ. Then, D is Lebesgue non-measurable.

Proof. Suppose the contrary, i. e. that D is Lebesgue measurable. First we shall prove that, for every interval U $(U \subset [0,1))$ and for almost all ξ,

$$\operatorname{fr}\{n: n\xi \in A_V \cap U (\mathrm{mod}\, 1)\} = \tfrac{1}{2}|U|, \tag{7}$$

where $|U|$ denotes the measure of U.

For brevity, we shall use the notations

$$\mathcal{W}^0 = \mathcal{W}, \quad \mathcal{W}^1 = \mathcal{W}', \quad V^0 = V \quad \text{and} \quad V^1 = V',$$

where $\mathcal{W}'$ denotes the complement of the set $\mathcal{W}$ to the set of all rationals and V' denotes the complement of the set V to the set of all positive ordinal numbers less than γ.

For every rational r $(r \neq 0)$ we denote by $k_n^{(i)}(r)$ $(n = 1, 2, \ldots)$ the sequence of naturals n such that $nr \in \mathcal{W}^i$ $(i = 0, 1)$.

It is well-known ([2], p. 344-346) that, for every sequence of integers $k_1 < k_2 < \ldots$ and for every interval U $(U \subset [0,1))$,

$$\operatorname{fr}\{n: k_n \xi \in U (\mathrm{mod}\, 1)\} = |U|$$

for almost all ξ. Consequently, for almost all ξ and for every rational r $(r \neq 0)$, the equality

$$\operatorname{fr}\{n: k_n^{(i)}(r)\xi \in U (\mathrm{mod}\, 1)\} = |U| \qquad (i = 0, 1). \tag{8}$$

From the definitions of the set A_V and the sequences $k_n^{(i)}(r)$ it follows directly that

$$\overline{\{n: n\xi \in A_V \cap U (\mathrm{mod}\, 1),\ n \leqslant N\}} = \overline{\{n: k_n^{(i)}(r(\xi))\xi \in U (\mathrm{mod}\, 1),\ k_n^{(i)}(r(\xi)) \leqslant N\}}$$

and

$$\overline{\{n: k_n^{(i)}(r(\xi)) \leqslant N\}} = \overline{\{n: nr(\xi) \in \mathcal{W}^i,\ n \leqslant N\}}$$

if $\alpha(\xi) \in V^i$ $(i = 0, 1)$. Hence,

$$\frac{1}{N}\overline{\{n: n\xi \in A_V \cap U (\mathrm{mod}\, 1),\ n \leqslant N\}} =$$

$$= \frac{1}{N}\overline{\{n: nr(\xi) \in \mathcal{W}^i,\ n \leqslant N\}} \cdot \frac{\overline{\{n: k_n^{(i)}(r(\xi))\xi \in U (\mathrm{mod}\, 1),\ k_n^{(i)}(r(\xi)) \leqslant N\}}}{\overline{\{n: k_n^{(i)}(r(\xi)) \leqslant N\}}}$$

if $\alpha(\xi) \in V^i$ $(i = 0, 1)$, which implies, in view of (8) and Lemma 1, the equality

$$\operatorname{fr}\{n : n\xi \in A_V \cap U \pmod 1\} =$$

$$= \operatorname{fr}\{n : nr(\xi) \in \mathcal{W}^i\} \operatorname{fr}\{n : k_n^{(i)}(r(\xi))\xi \in U \pmod 1\} = \tfrac{1}{2}|U| .$$

The formula (7) is thus proved.

From (6) and (7) it follows directly that, for every interval U and for almost all ξ, the following equality holds

$$\operatorname{fr}\{n : n\xi \in D \cap U \pmod 1\} = \tfrac{1}{2}|U| . \tag{9}$$

Further, in view of a Theorem of Raikov ([1], p. 377),

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} \chi(n\xi) - |D \cap U| \right| d\xi = 0 ,$$

where χ is the characteristic function of $D \cap U$ extended on the line with the period 1. Hence, and from (9), for every interval U, we obtain the equality $|D \cap U| = \frac{1}{2}|U|$, which contradicts the Lebesgue density theorem. The Lemma is thus proved.

Proof of Theorem 1. Let $V \in \mathfrak{V}$. By B_V we denote a set belonging to the base of the family Ξ such that

$$\operatorname{fr}\{n : n\xi \in A_V \dot{-} B_V \pmod 1\} = 0$$

for each irrational ξ. (According to Lemma 2 the sets A_V $(V \in \mathfrak{V})$ belong to Ξ). Applying Lemma 3 we find that the sets B_V are Lebesgue non-measurable. Since the power of the base is $\leqslant 2^{2^{\aleph_0}}$, then, to prove the Theorem, it suffices to show, in virtue of (5), that the function $V \to B_V$ establishes a one-to-one correspondence between sets V and sets B_V. Suppose $V_1 \neq V_2$. There is then an irrational ξ_0 such that $\alpha(\xi_0) \in V_1 \dot{-} V_2$. Taking into account the definition of A_V, we have $n\xi_0 \in A_{V_1} \dot{-} A_{V_2} \pmod 1$ $(n = 1, 2, \ldots)$. Hence, $\operatorname{fr}\{n : n\xi_0 \in A_{V_1} \dot{-} A_{V_2} \pmod 1\} = 1$, which implies $\operatorname{fr}\{n : n\xi_0 \in B_{V_1} \dot{-} B_{V_2} \pmod 1\} = 1$. Consequently, $B_{V_1} \neq B_{V_2}$.

Theorem 1 is thus proved.

Proof of Theorem 2. By $\mathfrak{V}_0$ we denote the class of all subsets of the set of all denumerable ordinal numbers. Obviously, $\mathfrak{V}_0 \subset \mathfrak{V}$ and $\bar{\mathfrak{V}}_0 = 2^{\aleph_1}$. By C_V $(V \in \mathfrak{V}_0)$ we denote a set belonging to the weak base of the family Ξ such that

$$\operatorname{fr}\{n : n\xi \in A_V \dot{-} C_V \pmod 1\} = 0$$

for almost all ξ. According to Lemma 3, the sets C_V are Lebesgue non-measurable. To prove the Theorem it suffices to show that the function: $V \to C_V$ $(V \in \mathfrak{V}_0)$ establishes a one-to-one correspondence between the

sets V and the sets C_V. Suppose $V_1 \neq V_2$ $(V_1, V_2 \in \mathfrak{V}_0)$. Similarly to the preceding proof we find that

$$\mathrm{fr}\{n : n\xi \in C_{V_1} \dot{-} C_{V_2} (\mathrm{mod}\, 1)\} = 1$$

for almost all ξ satisfying the condition $\alpha(\xi) \in V_1 \dot{-} V_2$. Obviously, to prove the inequality $C_{V_1} \neq C_{V_2}$ it is sufficient to show that the outer Lebesgue measure of the set $S = \{\xi : \alpha(\xi) \in V_1 \dot{-} V_2\}$ is positive. Suppose the contrary, i. e.

$$|S| = 0 . \tag{10}$$

Let η be the first ordinal number belonging to $V_1 \dot{-} V_2$. It is easy to verify that the real line R may be represented as the denumerable union of sets congruent to S

$$R = \bigcup_{\substack{r_1, \ldots, r_n \\ \alpha_1, \ldots, \alpha_n < \eta}} \left\{ x + \sum_{i=1}^{n} r_i x_{\alpha_i} : x \in S \right\},$$

where $r_1, \ldots, r_n$ are rationals $(n = 1, 2, \ldots)$. Hence, and from (10), it follows that $|R| = 0$, which is impossible. The Theorem is thus proved.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] D. A. Raikov, *On some arithmetical properties of summable functions* (in Russian), Matem. Sbornik **1** (1936), 377-384.

[2] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins.*, Math. Annalen **77** (1916), 313-352.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHEMATICS

On Independent Elements in Finitely Generated Algebras

by

S. ŚWIERCZKOWSKI

Presented by E. MARCZEWSKI on October 10, 1958

In this note we adopt the definitions and notations given by E. Marczewski [2]. Our purpose is to prove several theorems about independent elements in an algebra A which has a finite number of generators. We consider also the algebra $A^{(n)}$ (i. e. the algebra of algebraic operations of n variables on A), where A is arbitrary, and we prove a corollary about finite algebras $A^{(n)}$. Professor Marczewski pointed out that one of our results (at present Corollary 1) can be generalized. We start with the corresponding theorem.

THEOREM 1. *If a finite algebra A is generated by n elements and there exist n independent elements in A, then n elements $a_1, \ldots, a_n$ are generators of A if and only if they are independent.*

Proof. We assume that $a_1, \ldots, a_n$ are generators of A, and $b_1, \ldots, b_n$ are independent elements. We have to prove that $a_1, \ldots, a_n$ are independent and that $b_1, \ldots, b_n$ are generators. It follows from the independence of b_i that the mapping $b_i \to a_i$; $i = 1, \ldots, n$ can be extended to a homomorphism h of the subalgebra B generated by $b_1, \ldots, b_n$ on the subalgebra generated by $a_1, \ldots, a_n$, i. e. on A. Since A is finite, it follows from $B \subset A$, $h(B) = A$ that $A = B$ and that h is a $1:1$ mapping, i. e. an automorphism. Thus, $b_1, \ldots, b_n$ are generators of A and $a_1, \ldots, a_n$ are independent.

COROLLARY 1. *An algebra A, which is generated by n elements and contains $n+1$ independent elements, is infinite.*

Proof. Suppose that A is finite. Let $b_1, \ldots, b_{n+1}$ be independent elements. It follows from Theorem 1 that $b_1, \ldots, b_n$ generate A. But, clearly, b_{n+1} cannot be obtained by an algebraical operation from $b_1, \ldots, b_n$ and we arrive at a contradiction.

COROLLARY 2. *If, for some algebra A, the algebra $A^{(n)}$ is finite, then there exist no $n+1$ independent algebraical operations of n variables on A.*

Proof. This corollary is a consequence of Corollary 1, since $A^{(n)}$ is generated by the elements $e_i(x_1, \ldots, x_n) \equiv x_i$; $i = 1, \ldots, n$.

Example: A = a Boolean algebra, $A^{(n)}$ = the algebra of Boolean polynomials of n variables on A.

Theorem 2. *If an algebra A is generated by n independent elements and there exist also $n+1$ independent elements generating A, then there exist $n+2$ independent elements which generate A (consequently m independent generators for $m \geqslant n$).*

Proof. Let $a_1, \ldots, a_n$ be n independent generators and let $b_1, \ldots, b_{n+1}$ be $n+1$ independent generators of A. Thus, for some algebraical operations f_i, g_i,

$$b_i = f_i(a_1, \ldots, a_n); \quad i = 1, \ldots, n+1,$$
$$a_i = g_i(b_1, \ldots, b_{n+1}); \quad i = 1, \ldots, n.$$

We denote

$$(1) \quad \begin{aligned} h_i(x_1, \ldots, x_{n+1}) &\equiv f_i\big(g_1(x_1, \ldots, x_{n+1}), \ldots, g_n(x_1, \ldots, x_{n+1})\big), \\ k_i(x_1, \ldots, x_n) &\equiv g_i\big(f_1(x_1, \ldots, x_n), \ldots, f_{n+1}(x_1, \ldots, x_n)\big). \end{aligned}$$

Consequently, $h_i(b_1, \ldots, b_{n+1}) = b_i$ and $k_i(a_1, \ldots, a_n) = a_i$; from the independence of $a_1, \ldots, a_n$ and $b_1, \ldots, b_{n+1}$ follows

$$(2) \quad h_i(x_1, \ldots, x_{n+1}) \equiv x_i; \quad k_i(x_1, \ldots, x_n) \equiv x_i.$$

We now define $c_1, \ldots, c_{n+1}$ by $c_i = f_i(b_1, \ldots, b_n)$. Let us prove that $c_1, \ldots, c_{n+1}, b_{n+1}$ are independent generators of A. Indeed, it follows from

$$b_i = k_i(b_1, \ldots, b_n) = g_i(c_1, \ldots, c_{n+1})$$

that the algebra generated by $c_1, \ldots, c_{n+1}, b_{n+1}$ contains the generators $b_1, \ldots, b_{n+1}$ of A. To prove the independence of $c_1, \ldots, c_{n+1}, b_{n+1}$, let us suppose that, for some algebraical operations s, t, we have

$$(3) \quad s(c_1, \ldots, c_{n+1}, b_{n+1}) = t(c_1, \ldots, c_{n+1}, b_{n+1}).$$

We shall verify that, for arbitrary $d_1, \ldots, d_{n+2}$,

$$(4) \quad s(d_1, \ldots, d_{n+2}) = t(d_1, \ldots, d_{n+2}).$$

Evidently, (3) can be written in the form

$$(5) \quad \begin{aligned} &s\big(f_1(b_1, \ldots, b_n), \ldots, f_{n+1}(b_1, \ldots, b_n), b_{n+1}\big) = \\ &\qquad = t\big(f_1(b_1, \ldots, b_n), \ldots, f_{n+1}(b_1, \ldots, b_n), b_{n+1}\big). \end{aligned}$$

From the independence of b_i it follows that (5) will be preserved if we substitute for $b_1, \ldots, b_{n+1}$ arbitrary elements, e. g. d_{n+2} for b_{n+1} and $g_i(d_1, \ldots, d_{n+1})$ for b_i with $i \leqslant n$. It follows from (1) and (2) that, after this substitution, (5) passes into (4).

THEOREM 3. *If an n-generated algebra A contains $n+1$ independent elements, then A contains an infinite set of independent elements.*

EXAMPLE: A non-Abelian free group with n free generators.

Proof. We assume that $a_1, \ldots, a_n$ are the generators of A and that $b_1, \ldots, b_{n+1}$ are independent elements. Let $f_1, \ldots, f_{n+1}$ be these algebraical operations for which $b_i = f_i(a_1, \ldots, a_n)$.

We define by induction an infinite sequence of $n+1$-tuples

$$\langle c_1^j, \ldots, c_{n+1}^j \rangle; \qquad c_i^j \in A; \qquad j = 1, 2, \ldots$$

setting $c_i^1 = b_i$ and $c_i^{j+1} = f_i(c_1^j, \ldots, c_n^j)$. We assert that $c_{n+1}^1, c_{n+1}^2, \ldots$ are all independent (consequently different). Let us observe first that $c_1^m, \ldots, c_{n+1}^m$ are independent for every m. Indeed, this is true for $m = 1$. Suppose that $c_1^m, \ldots, c_{n+1}^m$ are independent for some m. Let C_m denote the subalgebra generated by $c_1^m, \ldots, c_n^m$. Then the mapping $c_i^m \to a_i$ $(i = 1, \ldots, n)$ can be extended to a homomorphism of C_m on A. Since the image of c_i^{m+1} is b_i, the independence of $b_1, \ldots, b_{n+1}$ implies the independence of $c_1^{m+1}, \ldots, c_{n+1}^{m+1}$.

To show the independence of $c_{n+1}^1, c_{n+2}^2, \ldots$ we ought to prove that if, for some m and for any algebraical operations of m variables g, h, the equality

$$g(c_{n+1}^1, \ldots, c_{n+1}^m) = h(c_{n+1}^1, \ldots, c_{n+1}^m) \tag{1}$$

holds, then $g = h$. Let us denote by $d_1, \ldots, d_m$ arbitrary elements of C_m. If we prove that (1) implies

$$g(d_1, \ldots, d_m) = h(d_1, \ldots, d_m), \tag{2}$$

then $g = h$ on C_m follows. But we observed above that C_m can be mapped homomorphically on A and thus $g = h$ on C_m implies $g = h$ on A. Instead of (2) let us prove by induction that

$$g(d_1, \ldots, d_l, c_{n+1}^{l+1}, \ldots, c_{n+1}^m) = h(d_1, \ldots, d_l, c_{n+1}^{l+1}, \ldots, c_{n+1}^m) \tag{3}$$

holds for $l = 1, \ldots, m$. We first prove (3) for $l = 1$. Since $C_{r+1} \subset C_r$ for every r, we obtain easily that $c_{n+1}^2, \ldots, c_{n+1}^m$ belong to C_1. Thus, there exist algebraical operations of n variables $s_2, \ldots, s_m$ such that

$$c_{n+1}^i = s_i(c_1^1, \ldots, c_n^1); \qquad i = 2, \ldots, m. \tag{4}$$

Substituting (4) into (1) we obtain

$$g\big(c_{n+1}^1, s_2(c_1^1, \ldots, c_n^1), \ldots, s_m(c_1^1, \ldots, c_n^1)\big) = \tag{5}$$
$$= h\big(c_{n+1}^1, s_2(c_1^1, \ldots, c_n^1), \ldots, s_m(c_1^1, \ldots, c_n^1)\big).$$

It follows from the independence of $c_1^1, \ldots, c_{n+1}^1$ that (5) will be preserved if we substitute d_1 for c_{n+1}^1 leaving $c_1^1, \ldots, c_n^1$ unchanged. After this substitution, (5) is transformed into (3) with $l = 1$.

Now suppose that (3) holds for some l and $1 \leqslant l < m$. It is clear that all elements $d_1, \ldots, d_l, c_{n+1}^{l+1}, \ldots, c_{n+1}^m$ belong to C_{l+1}. Thus, (3) leads to an equality of two algebraic operations on $c_1^{l+1}, \ldots, c_{n+1}^{l+1}$. Since $c_1^{l+1}, \ldots, c_{n+1}^{l+1}$ are independent, this equality will be preserved if we substitute d_{l+1} for c_{n+1}^{l+1} leaving $c_1^{l+1}, \ldots, c_n^{l+1}$ unchanged. Then we obtain (3) with d_{l+1} in place of c_{n+1}^{l+1}. Thus, (3) holds for $l = 1, \ldots, m$ and (2) follows.

Remark (added in proof). Some results connected with Theorem 1 are contained in [1]. There is contained also an example of an algebra satisfying the hypothesis of Theorem 2.

INSTITUTE OF MATHEMATICS (WROCŁAW BRANCH), POLISH ACADEMY OF SCIENCES

REFERENCES

[1] Jónsson and A. Tarski, *Two general theorems concerning free algebras*, Bull. Amer. Math. Soc. **62** (1956), 554.

[2] E. Marczewski, *A general scheme of the notions of independence in mathematics* (this issue).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

MATHEMATICS

Cohomotopic Multiplication and Duality Theorems Concerning Arbitrary Subsets of Euclidean Space

by

K. KURATOWSKI

Presented on October 17, 1958

1. Introduction. *). Let us denote: by $\mathcal{E}^n$ the n-dimensional Euclidean space ($n \geqslant 2$), by $\mathcal{S}_n$ the sphere $\mathcal{E}^n \cup (\infty)$, and by $\mathcal{P}_n$ the space $\mathcal{E}^n - (0)$. Let $\mathcal{X} \subset \mathcal{E}^n$. By $\mathcal{P}_n^{\mathcal{X}}$ we denote, as usually, the space of all continuous mappings of $\mathcal{X}$ into $\mathcal{P}_n$; the topology in this space is given by defining the closure of each $\Phi \subset \mathcal{P}_n^{\mathcal{X}}$:

$$(f \in \bar{\Phi}) \equiv (f|F \in \overline{\Phi|F} \text{ for each compact } F \subset \mathcal{X}) \text{ **}).$$

The topology thus defined, $\mathcal{P}_n^{\mathcal{X}}$ is a completely regular space. In the case of $\mathcal{X}$ compact (resp. locally compact), this topology is equivalent to the topology of uniform convergence (resp. uniform convergence on each compact set).

a and b being points of an arbitrary topological space A, we write $a \sim b$ to mean: A *is connected between* a *and* b, in other words: a and b belong to the same *quasi-component* of A (the quasi-component of a is the intersection of all closed-open subsets of A containing a). In general, the quasi-components need not to be connected, hence, they may not coincide with the components (= maximal connected subsets) of A. However, the quasi-components of $\mathcal{P}_n^{\mathcal{X}}$ (or, more generally, of $\mathcal{Y}^{\mathcal{X}}$, where $\mathcal{Y}$ is an absolute neighbourhood retract) are identical with their components [6], and thus, for $f_1, f_2 \in \mathcal{P}_n^{\mathcal{X}}$, we have

$$(f_1 \sim f_2) = (f_1 \text{ and } f_2 \text{ belong to the same component of } \mathcal{P}_n^{\mathcal{X}}).$$

*) This paper contains the essential points of my communication presented to the Int. Math. Congr. in Edinburgh, on August 15, 1958. The proofs of theorems stated here will be found in [7] and [3].

**) This topology is "compact-open". We recall that $f|F$ denotes the function f restricted to F; $\Phi|F$ denotes the set of all $f|F$, where $f \in \Phi$.

In the particular case, where $\mathfrak{X}$ is *compact* (or open in $\mathcal{E}^2$), the components of $\mathcal{P}_n^{\mathfrak{X}}$ are arcwise connected and therefore identical with *homotopy* classes ([3], p. 260, 3; this is no more the case for $\mathfrak{X}$ open in $\mathcal{E}^3$ [10] *)).

In what follows we shall consider first the *qualitative* part of duality theorems. In order to state a general duality theorem, we shall have to consider *the set* $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ *of all components of* $\mathcal{P}_n^{\mathfrak{X}}$ as a topological group. Dual to $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ will be the topological group $\mathfrak{N}(\mathcal{S}_n - \mathfrak{X})$ of integer-valued (normed) measures defined on closed-open subsets of $\mathcal{S}_n - \mathfrak{X}$. In **3** and **4** we shall state some fundamental properties of these groups; the theorems on $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$, as will be seen, are dual to those on $\mathfrak{N}(\mathcal{S}_n - \mathfrak{X})$.

2. Qualitative duality theorems. THEOREM 2.1. *The space* $\mathcal{P}_n^{\mathfrak{X}}$ *is connected if and only if the set* $\mathcal{S}_n - \mathfrak{X}$ *is connected* (for a direct proof see [6]).

This is an extension of a theorem of Borsuk on compact sets (for exact references, see [3], p. 343) to arbitrary sets $\mathfrak{X} \subset \mathcal{E}^n$.

The following statement is an analogue in n dimensions to a duality theorem of Eilenberg concerning the case $n = 2$ ([2], p. 90).

THEOREM 2.2. *Let* $p, q \in \mathcal{E}^n - \mathfrak{X}$. *Then*

$$(x-p \sim x-q \text{ in } \mathcal{P}_n^{\mathfrak{X}}) \equiv (p \sim q \text{ in } \mathcal{S}_n - \mathfrak{X}) \text{ **}) .$$

3. The topological group $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$. Let $\mathfrak{X} \subset \mathcal{E}^n$ and let us define the closure of $\mathfrak{Z} \subset \mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ by putting:

$$(\Gamma \in \bar{\mathfrak{Z}}) \equiv (\Gamma | F \in \mathfrak{Z} | F \text{ for each compact } F \subset \mathfrak{X}) \text{ ***}) .$$

It can be shown that the space $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ is *completely regular*.

In order to define the group-operation in $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$, denote by $\Gamma(f)$ the component of f in $\mathcal{P}_n^{\mathfrak{X}}$, and let us recall Borsuk's definition of cohomotopic multiplication (see [1], cf. also [9] and [4]). Given $f_1, f_2 \in \mathcal{P}_n^{\mathfrak{X}}$ such that, for each $x \in \mathfrak{X}$, either $f_1(x) = 1$ or $f_2(x) = 1$ (where 1 denotes the point $(1, 0, 0, 0, \ldots)$ of $\mathcal{E}^n$), the product $f_1(x) \cdot f_2(x)$ is well defined assuming the obvious rule $1 \cdot p = p = p \cdot 1$ (for each $p \in \mathcal{E}^n$); thus, in this particular case, the functions f_1 and f_2 are "multipliable". Now, by a theorem of Borsuk, to each pair $f_1, f_2 \in \mathcal{P}_n^{\mathfrak{X}}$ there exists a pair of multipliable functions f_1^*, f_2^* such that $f_1^* \in \Gamma(f_1)$ and $f_2^* \in \Gamma(f_2)$. Put

*) I am indebted for this reference to Professor J. H. C. Whitehead.

**) It is understood that the mappings $x-p$, $x-q$ etc. are restricted to $\mathfrak{X}$.

***) This is the "quotient topology" in respect to the relation $f_1 \sim f_2$.

$\Gamma(f_1)\cdot\Gamma(f_2)=\Gamma(f_1^*\cdot f_2^*)$. This definition is "correct", i. e. $\Gamma_1\cdot\Gamma_2$ does not depend upon the choice of multipliable functions f_1^* in Γ_1 and f_2^* in Γ_2 *).

The division $1:\Gamma$ is defined according to the rule:

$$\text{if} \quad p=(x_1, x_2, \ldots, x_n)\neq 0, \qquad 1:p=(x_1, x_2, \ldots x_{n-1}, -x_n):|p|^2.$$

In this way, $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ becomes a *topological Abelian group* (the multiplication and division being continuous) **).

It will be useful to write $f_0\sim f_1\cdot f_2$ instead of $\Gamma(f_0)=\Gamma(f_1)\cdot\Gamma(f_2)$ also in the case where f_1 and f_2 are not multipliable. The following theorem then holds true:

THEOREM 3.1. *Let $\mathfrak{X}$ be compact $\subset\mathcal{E}^n$. Then the components of $\mathcal{P}_n^{\mathfrak{X}}$ are closed-open, and therefore $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ is a discrete space. Let $R_0, R_1, \ldots$ be the sequence (finite or infinite) of components of $\mathcal{S}_n-\mathfrak{X}$ (R_0 containing the point ∞). Let $p_i\in R_i$ and $f\in\mathcal{P}_n^{\mathfrak{X}}$. Then we have:*

$$f(x)\sim(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_m)^{k_m},$$

where the exponents $k_1, \ldots, k_m$ are integers (which do not depend upon the choice of p_i in R_i).

Otherwise stated: *the components $\Gamma(x-p_1), \Gamma(x-p_2), \ldots$ are generators of the group $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$* ***).

For arbitrary $\mathfrak{X}\subset\mathcal{E}^n$, one has the following:

THEOREM 3.2. *Let $Q_0, Q_1, \ldots$ be a sequence of quasi-components of $\mathcal{S}_n-\mathfrak{X}$ such that $\mathcal{S}_n-\mathfrak{X}\subset\overline{Q_0\cup Q_1\cup\ldots}$ (Q_0 containing ∞). Put $p_i\in Q_i$. Then the group of "rational components", i. e. of components of the form $\Gamma^{k_1}(x-p_1)\cdot\ldots\cdot\Gamma^{k_m}(x-p_m)$, is dense in $\mathfrak{L}(\mathcal{P}^{\mathfrak{X}})$. Hence, $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ is separable.*

THEOREM 3.3. *If $\mathfrak{X}$ is locally compact, $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ is metric, complete and 0-dimensional (and therefore homeomorphic to a closed subset of the space of irrational numbers)* [5]. *Hence, each component of $\mathcal{P}_n^{\mathfrak{X}}$ is the limit of a sequence of rational components.*

In particular, if $\mathfrak{X}$ is open $\subset\mathcal{E}^2$, each $f\in\mathcal{P}_2^{\mathfrak{X}}$ is of the form

$$f(x)=\lim_{i=\infty} r_i(x)\cdot e^{q_i(x)},$$

---

*) This theorem was shown by Borsuk for the case of $\mathfrak{X}$ compact $\subset\mathcal{E}^n$ and $n\geqslant 3$ (see [1] and [9]). However, it can be easily extended to arbitrary $\mathfrak{X}\subset\mathcal{E}^n$ for $n\geqslant 2$ (see [3]).

**) For $\mathfrak{X}$ compact, $\mathfrak{L}(\mathcal{S}_{n-1}^{\mathfrak{X}})$ is the Borsuk's $n-1$ *cohomotopy group*.

***) For exact references see [4], p. 283.

where r_i is rational and g_i continuous, the convergence being uniform on each compact subset of $\mathfrak{X}$ *).

THEOREM 3.4. *Let $\mathfrak{X}$ be open and let $\mathcal{S}_n - \mathfrak{X}$ contain infinitely many components; then the topological groups $\mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ and $\mathcal{G}^{\aleph_0}$ (the group of all infinite sequence of integers) are isomorphic.*

THEOREM 3.5. *Let $\mathfrak{X}$ be open and suppose the components of $\mathcal{Y} = \mathcal{S}_n - \mathfrak{X}$ to form an infinite sequence $Q_0, Q_1, \ldots$, where each Q_i is open in $\mathcal{Y}$, except Q_0 which contains ∞* (in the case of an entire function, $Q_1, Q_2, \ldots$ are its zeros and Q_0 its pole). *Then each $\Gamma \in \mathfrak{L}(\mathcal{P}_n^{\mathfrak{X}})$ can be represented in the form*

$$\Gamma = \prod_{i=1}^{\infty} \Gamma^{k_i}(x - p_i) .$$

Again, in the case of $n = 2$, this can be stated more precisely: each $f \in \mathcal{P}_n^{\mathfrak{X}}$ is of the form

$$f(x) = \prod_{i=1}^{\infty} (x - p_i)^{k_i} e^{g_i(x)} ,$$

the convergence being uniform on each compact subset of $\mathfrak{X}$.

4. The topological group $\mathfrak{N}(\mathcal{Y})$. Let $\infty \in \mathcal{Y} \subset \mathcal{S}_n$. Denote by $(0,1)^{\mathcal{Y}}$ the set of all closed-open subsets of $\mathcal{Y}$, and by $\mathfrak{N}(\mathcal{Y})$ **) the set of all integer-valued functions μ (called measures), defined on elements Z of $(0,1)^{\mathcal{Y}}$, additive and normed, i. e.

1^0 *if* $Z = \bigcup_i Z_i$ *where* $Z_i \cap Z_{i'} = 0$ *for* $i \neq i'$, *then* $\mu(Z) = \sum_i \mu(Z_i)$,

2^0 $\mu(\mathcal{Y}) = 0$.

The topology in $\mathfrak{N}(\mathcal{Y})$ is defined as follows. Call $\boldsymbol{C}$-set any subset $\boldsymbol{C}$ of $(0,1)^{\mathcal{Y}}$, containing a family of disjoint sets $\{B_i\}$ (the base of $\boldsymbol{C}$) such that: $Z \in \boldsymbol{C}$ if and only if Z is the union of some of the sets B_i; moreover, let $\mathcal{Y} \in \boldsymbol{C}$. Now for each $\mathfrak{X} \subset \mathfrak{N}(\mathcal{Y})$ we define its closure by putting:

$$(\mu \in \overline{\mathfrak{X}}) \equiv [(\mu|\boldsymbol{C}) \in (\mathfrak{X}|\boldsymbol{C}) \quad \textit{for each} \quad \boldsymbol{C}\textit{-set}] .$$

With this definition, $\mathfrak{N}(\mathcal{Y})$ *is a completely regular space.*

As the addition $\mu_0 = \mu_1 + \mu_2$ (which means $\mu_0(Z) = \mu_1(Z) + \mu_2(Z)$ for each Z) is continuous, $\mathfrak{N}(\mathcal{Y})$ is a free Abelian topological group.

p being a point $\neq \infty$ of $\mathcal{Y}$, we call characteristic normed measure β_p of p, the measure $\beta_p \in \mathfrak{N}(\mathcal{Y})$ such that for each Z we have: $\beta_p(Z) = 1$ if $p \in Z$ and $\infty \in \mathcal{Y} - Z$, and $\beta_p(Z) = 0$ if $p, \infty \in Z$.

*) The latter part of Theorem 3.3 corresponds to the well known Runge theorem. Theorem 3.5 corresponds to the Weierstrass decomposition theorem on entire functions (see [3], p. 410).

**) The definition of $\mathfrak{N}(\mathcal{Y})$ may be easily extended to arbitrary topological spaces.

Then we have:

THEOREM 4.1. *Let $\mathcal{Y}$ be locally connected and let $R_0, R_1, \ldots$ be the sequence of its components ($\infty \in R_0$). Then this sequence is a base of $(0,1)^{\mathcal{Y}}$ and, consequently, $\mathfrak{N}(\mathcal{Y})$ is discrete. Put $p_i \in R_i$. Then $\beta_{p_1}, \beta_{p_2} \ldots$ are generators of the group $\mathfrak{N}(\mathcal{Y})$.*

THEOREM 4.2. *If $\mathcal{Y}$ is arbitrary and $Q_0, Q_1, \ldots$ satisfy the hypotheses of Theorem 3.2 (with $\mathcal{Y} = \mathcal{S}_n - \mathcal{X}$), the group $\mathfrak{R}$ generated by the measures β_{p_i} is dense in $\mathfrak{N}(\mathcal{Y})$. Thus $\mathfrak{N}(\mathcal{Y})$ is separable.*

THEOREM 4.3. *If $\mathcal{Y}$ is the union of a compact set and a set open in $\mathcal{S}_n$, then $\mathfrak{N}(\mathcal{Y})$ is metric, complete, 0-dimensional, and each $\mu \in \mathfrak{N}(\mathcal{Y})$ is limit of elements of $\mathfrak{R}$.*

THEOREM 4.4. *Let $\mathcal{Y}$ be compact and contain an infinity of components. Then the topological groups $\mathfrak{N}(\mathcal{Y})$ and $\mathcal{G}^{\aleph_0}$ are isomorphic* ([3], p. 418).

THEOREM 4.5. *Let $\mathcal{Y}$ be compact and satisfy the hypotheses of Theorem 3.5. Then each $\mu \in \mathfrak{N}(\mathcal{Y})$ may be developed in a series:*

$$\mu = \sum_{i=1}^{\infty} k_i \mu_i, \quad \text{where} \quad k_i = \mu(Q_i) \quad \text{and} \quad \mu_i = \beta_{p_i}.$$

5. General duality theorem. *For each $\mathcal{X} \subset \mathcal{E}^n$ there exists an (homeomorphic) isomorphism μ of $\mathfrak{L}(\mathcal{P}_n^{\mathcal{X}})$ into $\mathfrak{N}(\mathcal{S}_n - \mathcal{X})$. If $\mathcal{X}$ is locally compact, μ is onto.*

Moreover: $\mu\Gamma(x-p) = \beta_p$ for each $p \in \mathcal{S} - \mathcal{X}$.

μ is defined in the following way. Let us first consider the case, where $\mathcal{X}$ is compact. Using the notations of Theorem 3.1. put $\mu_{R_i} f = k_i$. If $Z \in (0,1)^{\mathcal{Y}}$, where $\mathcal{Y} = \mathcal{S}_n - \mathcal{X}$, Z is of the form $Z = R_{i_1} \cup R_{i_2} \cup \ldots$, and we put $\mu_Z f = k_{i_1} + k_{i_2} + \ldots$

If $\mathcal{X}$ is arbitrary, set $\mu_Z f = \mu_H(f, F)$, where F is compact and $H \in (0,1)^{\mathcal{S}_n - F}$ such that $Z = H - \mathcal{X}$ (the existence of sets F and H can be easily shown, and the value of $\mu_Z f$ does not depend upon their choice). Finally, for $\Gamma \in \mathfrak{L}(\mathcal{P}_n^{\mathcal{X}})$ we put $\mu_Z \Gamma = \mu_Z f$, where $f \in \Gamma$. Thus $\mu_Z \Gamma$ is a well defined integer, and $(\mu\Gamma) \in \mathfrak{N}(\mathcal{S}_n - \mathcal{X})$ *); μ is the required isomorphism.

The following statement is a generalization of Theorem 2.2.

Let $Q_0, \ldots, Q_m$ be different quasi-components of $\mathcal{S}_n - \mathcal{X}$, and $\infty \in Q_0$. Let $p_i \in Q_i$. Then the components $\Gamma(x-p_1), \ldots, \Gamma(x-p_m)$ of $\mathcal{P}_n^{\mathcal{X}}$ are linearly independent.

For, as may easily be seen, $\beta_{p_1}, \ldots, \beta_{p_m}$ are linearly independent.

In the particular case, where $\mathcal{S}_n - \mathcal{X}$ contains but a finite number of components, say $m+1$, the group $\mathfrak{L}(\mathcal{P}_n^{\mathcal{X}})$ (as well as $\mathfrak{N}(\mathcal{S}_n - \mathcal{X})$) is

*) Thus μf is an extension of Brouwer's degree to the case of arbitrary $\mathcal{X} \subset \mathcal{E}^n$.

isomorphic to $\mathcal{G}^m$. This gives another proof of invariance of the number of components of $\mathcal{S}_n - \mathcal{X}$ (assuming this number to be finite).

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] K. Borsuk, *Sur les groupes des classes de transformations continues*, C. R. Paris **202** (1936), 1400.

[2] S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), 61-112.

[3] K. Kuratowski, *Topologie* vol. II, Monogr. Mat., new edition 1959.

[4] — *Sur l'extension de la notion de fonction rationelle à l'espace euclidien n-dimensionnel*, Bull. Acad. Polon. Sci. Sér. des sci. math., astr. et phys. **6** (1958), 281-287.

[5] — *Sur les composantes de l'espace des transformations d'un espace localement compact en un rétracte de voisinage*, ibid. **6** (1958), 565-571.

[6] — to appear in Math. Zeitschrift 1959.

[7] — to appear in Fund. Math. 1959.

[8] K. Morita, *Cohomotopy groups in fully normal spaces*, Science Reports, Tokyo, Bunrika Daigaku **4** (1953), 251-261.

[9] E. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. **50** (1949), 203-245.

[10] N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. of Math. **41** (1940), 833-851.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

On the Canonical Formalism for Dynamical Systems with Constraints

by

A. PAWLIKOWSKI and W. SZCZURÓWNA

Presented by W. RUBINOWICZ on September 30, 1958

1. Let us consider a dynamical system with n degrees of freedom, described by $n+\nu$ generalized co-ordinates q_i, Q_α *). The co-ordinates should obey ν equations of holonomic constraints. Suppose these equations to be of the form

$$Q_\alpha = \varphi_\alpha(q) \quad **) . \tag{1}$$

We assume the Lagrangian of the system $L(q, \dot{q}, Q, \dot{Q})$ to be a quadratic form of all $\dot{q}_i$. It is well-known that the equations of motion and the equations of constraints can be obtained as Euler-Lagrange equations (*E-L* Eqs.) of the variational problem

$$\delta \int_{t_0}^{t_1} \{L + \varkappa_\alpha(Q_\alpha - \varphi_\alpha)\}\, dt = 0 \quad ***) , \tag{2}$$

where the variables to be varied are $q_i, Q_\alpha, \varkappa_\alpha$.

The commonly used transition to canonical formalism is possible only in the case when the Lagrangian is such a function of $\dot{Q}_\alpha$ that equations

$$P_\alpha = \frac{\partial L}{\partial \dot{Q}_\alpha}$$

determine the $\dot{Q}_\alpha$ as functions of generalized co-ordinates and their conjugate momenta. Then, the motion of the system is determined by the canonical equations

$$\dot{q}_i = \frac{\partial \overline{\mathcal{H}}}{\partial P_i}, \quad \dot{p}_i = -\frac{\partial \overline{\mathcal{H}}}{\partial q_i}, \quad \dot{Q}_\alpha = \frac{\partial \overline{\mathcal{H}}}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial \overline{\mathcal{H}}}{\partial Q_\alpha}, \tag{3}$$

*) In 1 index i runs from 1 to n, and index α runs from 1 to ν.

**) We write, for the sake of shortness, $F(q, \dot{q}, Q, \dot{Q})$ for functions of $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, Q_1, \ldots, Q_\nu, \dot{Q}_1, \ldots, \dot{Q}_\nu$.

***) We use the Einstein summation convention over repeated indices.

where

$$\bar{\mathcal{H}} = \mathcal{H} - \varkappa_a(Q_a - \varphi_a), \qquad \mathcal{H} = -L + \dot{q}_i p_i + \dot{Q}_a P_a, \tag{4}$$

and by the equations of constraints (1).

In this note we give a method of transition to canonical formalism which holds in the case when the Lagrangian depends on $\dot{Q}_a$, in the manner mentioned above, as well as in the case when the Lagrangian does not depend on $\dot{Q}_a$. This is accomplished as follows. We add the term $\frac{d}{dt}(\sigma_a(Q_a - \varphi_a))$, (where $\sigma_a = -\int^t \varkappa_a dt$) to the integrand in (2) and obtain the variational problem

$$\delta \int_{t_0}^{t_1} \{L + \sigma_a(\dot{Q}_a - \dot{\varphi}_a)\} dt = 0, \tag{5}$$

which is equivalent to (2) if we assume the initial conditions

$$Q_a|_{t=t_0} = \varphi_a|_{t=t_0}. \tag{6}$$

We perform the transition from (5) to the canonical formalism using the well-known method of [1]. Following this method we introduce $\dot{q}_i$ and $\dot{Q}_a$ as independent variables. The equations of motion and the equations of constraints are obtained at the E-L Eqs. of the following variational problem

$$\delta \int_{t_0}^{t_1} \{L(q, k, Q, K) + \lambda_i(\dot{q}_i - k_i) + \Lambda_a(\dot{Q}_a - K_a) + \sigma_a(K_a - \varphi_{a,i} k_i)\} dt = 0, \tag{7}$$

where the variables to be varied are $q_i, k_i, Q_a, K_a, \lambda_i, \Lambda_a, \sigma_a$, and $\varphi_{a,i}$ denotes $\frac{\partial \varphi_a}{\partial q_i}$. These equations are:

$$\text{a)} \quad \frac{\partial L}{\partial q_i} - \sigma_a \varphi_{a,ij} k_j - \dot{\lambda}_i = 0, \tag{8}$$

$$\text{b)} \quad \frac{\partial L}{\partial k_i} - \lambda_i - \sigma_a \varphi_{a,i} = 0,$$

$$\text{c)} \quad \frac{\partial L}{\partial Q_a} - \dot{\Lambda}_a = 0,$$

$$\text{d)} \quad \frac{\partial L}{\partial K_a} - \Lambda_a + \sigma_a = 0,$$

$$\text{e)} \quad \dot{q}_i - k_i = 0,$$

$$\text{f)} \quad \dot{Q}_a - K_a = 0,$$

$$\text{g)} \quad K_a - \varphi_{a,i} k_i = 0.$$

Now, instead of the Eqs. (8 e, f), we may assume Eqs. (8 b, d) to be equations of constraints. Let us consider first the case when variables

k_i and K_a can be determined from Eqs. (8 b, d) as functions of q_i, λ_i, Q_a, Λ_a:

$$k_i = k_i(q, \lambda, Q, \Lambda)\,, \qquad K_a = K_a(q, \lambda, Q, \Lambda)\,. \tag{9}$$

The variables λ_i and Λ_a are the momenta canonically conjugate to q_i and Q_a, respectively. We substitute (9) into the integrand in (7), and obtain

$$\delta \int_{t_0}^{t_1} \{\ \}\, dt = \delta \int_{t_0}^{t_1} \{\lambda_i \dot{q}_i + \Lambda_a \dot{Q}_a - \overline{\mathcal{H}}(q, \lambda, Q, \Lambda)\}\, dt = 0\,, \tag{10}$$

where

$$\overline{\mathcal{H}} = \mathcal{H} - \sigma_a(K_a - \varphi_{a,i} k_i) \tag{11}$$

and

$$\mathcal{H} = -L(q, k, Q, K) + \lambda_i k_i + \Lambda_a K_a\,. \tag{12}$$

The E-L Eqs. of the variational problem (10) are the canonical equations

$$\dot{q}_i = \frac{\partial \overline{\mathcal{H}}}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial \overline{\mathcal{H}}}{\partial q}, \quad \dot{Q}_a = \frac{\partial \overline{\mathcal{H}}}{\partial \Lambda_a}, \quad \dot{\Lambda}_a = -\frac{\partial \overline{\mathcal{H}}}{\partial Q_a}, \tag{13}$$

and the equations of constraints

$$K_a = \varphi_{a,i} k_i\,. \tag{14}$$

Obviously, we must add to these equations the initial conditions (6).

Let us consider now the case when the Lagrangian in (7) does not depend on $\dot{Q}_a$. Then, the Eqs. (8d) take the form

$$\Lambda_a = \sigma_a\,, \tag{15}$$

and, therefore, the variables K_a cannot be determined from them as functions of q_i, λ_i, Q_a, Λ_a. Taking into account (15), we obtain from (7)

$$\delta \int_{t_0}^{t_1} \{\ \}\, dt = \delta \int_{t_0}^{t_1} \{\lambda_i \dot{q}_i + \Lambda_a \dot{Q}_a - H(q, \lambda, Q, \Lambda)\}\, dt = 0\,, \tag{16}$$

where

$$H = -L(q, k, Q) + \lambda_i k_i + \Lambda_a \varphi_{a,i} k_i\,. \tag{17}$$

The E-L Eqs. of this variational problem, i. e. the corresponding canonical equations, are:

$$\begin{aligned} &\text{a)} \quad \dot{q}_i = \frac{\partial H}{\partial \lambda_i}, \qquad \text{c)} \quad \dot{Q}_a = \frac{\partial H}{\partial \Lambda_a}, \\ &\text{b)} \quad \dot{\lambda}_i = -\frac{\partial H}{\partial q_i}, \qquad \text{d)} \quad \dot{\Lambda}_a = -\frac{\partial H}{\partial Q_a}, \end{aligned} \tag{18}$$

to which the initial conditions (6) are added. It is noteworthy that the Eqs. (18c) are equations of constraints.

Notice that the general dynamical problem with constraints of type (1) can always be reduced to the case when the Lagrangian does not depend on $\dot{Q}_\alpha$. As may be seen, the equations of constraints enable us to eliminate all $\dot{Q}_\alpha$ form the Lagrangian. Therefore, (18) can be considered as a general form of canonical Hamilton equations for dynamical systems with constraints of type (1).

2. The above result may be applied to the collective variables theory which has been widely used recently in various branches of physics. Namely, we can give the classical variant of the collective variables theory of Bohm-Pines [2] and Migdał-Galitsky's [3], [4] type.

Consider a system of N identical particles interacting through the two-body potential $W(|\vec{x}_j - \vec{x}_l|)$. The Lagrangian of the system is

$$L = \sum_{j=1}^{N} \frac{m\dot{\vec{x}}_j^2}{2} - \frac{1}{2} \sum_{\substack{j,l=1 \\ j \neq 1}}^{N} W(|\vec{x}_j - \vec{x}_l|) \,. \tag{19}$$

Following the notation of **1** we shall also denote $x_1^1 = q_1$, $x_1^2 = q_2$, $x_1^3 = q_3$, $x_2^1 = q_4$, $x_N^3 = q_{3N}$. We write the potential energy in the form of the Fourier series and decompose it into two terms, as follows

$$\tfrac{1}{2} \sum_{j \neq l} W(|\vec{x}_j - \vec{x}_l|) = V'(q) + V''\big(\varrho(q)\big) \,, \tag{20}$$

where

$$V'(q) = -\tfrac{1}{2} N W(0) + \tfrac{1}{2} \sum_{|\vec{k}| > k_0} \sum_{j,l=1}^{N} W_k e^{i\vec{k}(\vec{x}_j - \vec{x}_l)} \,, \text{ *)} \tag{21}$$

$$V''\big(\varrho(q)\big) = \tfrac{1}{2} \sum_{|\vec{k}| < k_0} W_k \varrho_{\vec{k}} \varrho_{-\vec{k}} \tag{22}$$

and

$$\varrho_{\vec{k}}(q) = \sum_{j=1}^{N} e^{-i\vec{k}\vec{x}_j} \,. \tag{23}$$

We now consider two dynamical problems equivalent to the problem considered above and we apply to them the reasoning of **1**.

I. The Lagrangian is

$$L^{\mathrm{I}} = \sum_{i=1}^{3N} \frac{m\dot{q}_i^2}{2} - V'(q) - V''(\varrho + Q) \,, \tag{24}$$

and the constraints imposed on the system are

$$Q_{\vec{k}} = 0 \,, \quad (|\vec{k}| < k_0) \text{ **)}. \tag{25}$$

*) We assume that the particles are contained in a unit volume.

) The index α from **1 corresponds here to the vectorial index $\vec{k}(|\vec{k}| < k_0)$.

Hence, we have

$$\varphi_{\vec{k}} = 0 . \tag{26}$$

We substitute (24) and (26) into (17) and we obtain the Hamiltonian

$$H^{\mathrm{I}} = \sum_{i=1}^{3N} \frac{\lambda_i^2}{2m} + V'(q) + V''(\varrho + Q) . \tag{27}$$

II. In this case the Lagrangian is

$$L^{\mathrm{II}} = \sum_{i=1}^{3N} \frac{m\dot{q}_i^2}{2} - V'(q) - V''(Q) , \tag{28}$$

and the constraints are

$$Q_{\vec{k}} = \varrho_{\vec{k}} \qquad (|\vec{k}| < k_0) , \tag{29}$$

that is,

$$\varphi_{\vec{k}} \equiv \varrho_{\vec{k}} \qquad (|\vec{k}| < k_0) . \tag{30}$$

Substituting (28) and (30) into (17) we obtain

$$H^{\mathrm{II}} = \sum_{i=1}^{3N} \frac{1}{2m} \left(\lambda_i + \sum_{|\vec{k}|<k_0} \Lambda_{\vec{k}} \varrho_{\vec{k},i} \right)^2 + V'(q) + V''(Q) . \tag{31}$$

There exists a canonical transformation which enables us to pass from H^{I} to H^{II}. The generating function of this transformation is

$$\Omega(q', Q', \lambda, \Lambda) = \sum_{|\vec{k}|<k_0} \Lambda_{\vec{k}} \left(Q'_{\vec{k}} + \varrho_{\vec{k}}(q') \right) + \sum_{i=1}^{3N} q_i \lambda_i , \tag{32}$$

where the primed variables are the ones from H^{I}.

The Hamiltonians H^{I} and H^{II} are classical analogies of the extended Hamiltonian and of the basic collective Hamiltonian of [2], respectively.

DEPARTMENT OF THEORETICAL PHYSICS, WROCŁAW UNIVERSITY

REFERENCES

[1] K. Nordheim, E. Fuess, *Handbuch der Physik*, Vol. 5, Berlin, 1927, p. 92-95.
[2] D. Pines, *Solid state physics*, Vol. 1, 1955, p. 368.
[3] D. N. Zubarev, Candidate's degree dissertation, Moscow, 1953.
[4] Z. Galasiewicz, Postępy Fizyki, **7** (1956), 317.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

Über den Zusammenhang der vom Feld abhängigen Eigenmasse mit dem konform-Minkowskischen Raume

von

C. JANKIEWICZ

Vorgelegt von L. INFELD am 2 Oktober 1958

Einleitung

G. Szamosi [1] und L. Infeld [2] haben gezeigt, dass man die Bewegungsgleichungen für ein Teilchen mit einer vom Feld abhängigen Eigenmasse auch durch eine Verallgemeinerung der Lagrangeschen Funktion der speziellen Relativitätstheorie erhalten kann. In der vorliegenden Note wird gezeigt, dass diese Gleichungen die Gleichungen einer geodätischen Linie im konform-Minkowskischen Raume sind. Mit der Annahme, dass die skalare Krümmung des Raumes gleich Null ist, wird die Poissonsche Gleichung für das Feld abgeleitet.

Der konform-Minkowskische Raum

Der Raum V_4 heisst ein konform-Minkowskischer C_4 wenn sein metrischer Tensor $g'_{\alpha\beta}$ dem des Minkowskischen Raumes $g_{\alpha\beta}$ proportional ist

$$g'_{\alpha\beta} = \psi^2 g_{\alpha\beta}\,, \tag{1}$$

wo ψ eine beliebige Funktion des Punktes x^α ist. (Die griechischen Indizes nehmen die Werte 1, 2, 3, 4 an; Indizes, die sich übereinander wiederholen, sollen von 1 bis 4 summiert werden). Die Bogenelemente entsprechender Räume haben die Form

$$ds'^2 = g'_{\alpha\beta} dx^\alpha dx^\beta\,, \qquad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta\,. \tag{2}$$

Aus (1) und (2) folgt die Relation

$$ds' = \psi ds\,. \tag{3}$$

Die Christoffelschen Symbole des Raumes C_4 gibt die Gleichung

$$\nabla'_\alpha g'_{\mu\nu} = 0\,. \tag{4}$$

Bei Beachtung von (1) und (4) bekommen wir

$$(5)\qquad \Gamma'^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \omega_\alpha \delta^\gamma_\beta + \omega_\beta \delta^\gamma_\alpha - \omega_\nu g^{\nu\gamma} g_{\alpha\beta}\,, \qquad \Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\varrho}\left(\frac{\partial g_{\alpha\varrho}}{\partial x^\beta} + \frac{\partial g_{\beta\varrho}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\varrho}\right),$$

wo

$$(6)\qquad \omega_\mu = \psi^{-1}\partial_\mu \psi\,.$$

Aus der Formel (1) folgt ausserdem die Relation

$$(7)\qquad \sqrt{-g'} = \psi^4 \sqrt{-g}\,.$$

Der Krümmungstensor des Raumes C_4 hat die Gestalt (s. z. B. [3])

$$(8)\qquad R'_{\mu\alpha\beta}{}^{\nu} = g^{\sigma\nu} g_{[\sigma[\alpha}\omega_{\mu]\beta]}\,,$$

wo

$$(9)\qquad \omega_{\alpha\beta} = V_\alpha \omega_\beta - \omega_\alpha \omega_\beta + \tfrac{1}{2} g^{\mu\nu}\omega_\mu \omega_\nu g_{\alpha\beta}\,.$$

Aus (8) bekommen wir

$$(10)\qquad R'_{\alpha\beta} = -2\omega_{\alpha\beta} - g^{\mu\nu}\omega_\mu \omega_\nu g_{\alpha\beta}\,.$$

Die skalare Krümmung ist durch die Formel

$$(11)\qquad R' = -6\psi^{-2} g^{\mu\nu}\omega_{\mu\nu}$$

gegeben.

Feld- und Bewegungsgleichungen im konform-Minkowskischen Raume

Die Metrik des konform-Minkowskischen Raumes hängt von einer willkürlichen Funktion ab, die wir als ψ-Feld bezeichnen werden. Die Bewegung seiner Singularitäten werden wir durch die Weltlinien $\xi^\alpha(s')$ darstellen. Alle Gleichungen werden wir aus dem Variationsprinzip herleiten.

Die Wirkungsfunktion für das ψ-Feld und seine Singularitäten nehmen wir in der nachstsehender Form an

$$(12)\qquad S = \int\left[\frac{3\gamma}{\pi}\int \delta_4(x^\gamma - \xi'^\gamma)\sqrt{g'_{\alpha\beta}\frac{d\xi^\alpha}{ds'}\frac{d\xi^\beta}{ds'}}\,ds' + R'\sqrt{-g'}\right](dx),\quad \xi'^\gamma = \xi^\gamma(s')\,.$$

Das erste Glied in (12) stellt ein Integral über die Eigenzeit des Raumes C_4 dar

$$(13)\qquad S_{\mathrm{I}} = \int ds', \qquad (x^\alpha = \xi'^\alpha)\,,$$

das zweite Glied ein Integral der skalaren Krümmung über ein invariantes Raumvolumen

$$(14)\qquad S_{\mathrm{II}} = \int R'\sqrt{-g'}\,(dx)\,.$$

Die Variation von (12) nach dem Tensor $g'_{\alpha\beta}$ hat die Gestalt

$$\delta S = \int \left[\frac{3\gamma}{2\pi} \int \delta_4(x^\gamma - \xi'^\gamma) \frac{d\xi^\alpha}{ds'} \frac{d\xi^\beta}{ds'} ds' - \left(R'^{\alpha\beta} - \frac{1}{2} R' g'^{\alpha\beta} \right) \sqrt{-g'} \right] \delta g'_{\alpha\beta}(dx) . \tag{15}$$

Aus (1) folgt

$$\delta g'_{\alpha\beta} = 2\psi^{-1} g'_{\alpha\beta} \delta\psi . \tag{16}$$

Wenn wir (16) in (15) einsetzen, erhalten wir

$$\delta S = \int \left[\frac{3\gamma}{\pi} \int \delta_4(x^\gamma - \xi'^\gamma) ds' + 2R' \sqrt{-g'} \right] \psi^{-1} \delta\psi(dx) . \tag{17}$$

Bei Beachtung der Willkür der Variation $\delta\psi$ bekommen wir aus der Bedingung $\delta S = 0$ die Feldgleichung

$$R' = -\frac{3\gamma}{2\pi \sqrt{-g'}} \int \delta_4(x^\mu - \xi'^\mu) ds' . \tag{18}$$

Diese Gleichung zeigt, dass die skalare Krümmung des Raumes C_4 überall ausserhalb der Singularitätenweltlinien $\xi^\mu(s')$ gleich Null ist.

Die Bewegungsgleichungen der ψ-Feldsingularitäten folgen aus der Bedingung

$$\delta S = \delta S_{\mathrm{I}} = 0 . \tag{19}$$

Wenn wir diese Variation an (13) ausführen, bekommen wir den Ausdruck

$$\delta S_{\mathrm{I}} = \int \frac{3\gamma}{\pi} \left(\frac{d^2\xi^\alpha}{ds'^2} + \Gamma'^{\alpha}_{\mu\nu} \frac{d\xi^\mu}{ds'} \frac{d\xi^\nu}{ds'} \right) g'_{\alpha\beta} \delta\xi^\beta ds' = 0 \tag{20}$$

und daraus die Bewegungsgleichungen

$$\frac{d^2\xi^\alpha}{ds'^2} + \Gamma'^{\alpha}_{\mu\nu} \frac{d\xi^\mu}{ds'} \frac{d\xi^\nu}{ds'} = 0 . \tag{21}$$

Sie zeigen, dass die ψ-Feldsingularitäten sich entlang der geodätischen Linien des Raumes C_4 bewegen.

Feld- und Bewegungsgleichungen im Minkowskischen Raume

Bei Beachtung von (6) und (9) erhalten wir

$$\omega_\alpha \omega_\beta = \psi^{-2} \psi_\alpha \psi_\beta, \nabla_\alpha \omega_\beta = -\psi^{-2} \psi_\alpha \psi_\beta + \psi^{-1} \nabla_\alpha \psi_\beta , \tag{22}$$

wo

$$\psi_\alpha = \nabla_\alpha \psi = \partial_\alpha \psi . \tag{23}$$

Durch das Einsetzen von (22) in (11) erhält man

$$R' = -6\psi^{-3} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \psi . \tag{24}$$

Aus (3) folgt

$$\int \delta_4(x^\mu - \xi'^\mu)\,ds' = \int \psi \delta_4(x^\mu - \xi^\mu)\,ds\,. \tag{25}$$

Mit (7), (24) und (25) bekommen wir die Poissonsche Gleichung für das ψ-Feld

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \psi = \frac{1}{4\pi} \frac{\gamma}{\sqrt{-g}} \int \delta_4(x^\mu - \xi^\mu)\,ds\,. \tag{26}$$

Jetzt untersuchen wir die Bewegungsgleichungen (21). Wegen (5) können wir schreiben

$$\frac{d^2\xi^\nu}{ds'^2} + \Gamma^\nu_{\alpha\beta} \frac{d\xi^\alpha}{ds'} \frac{d\xi^\beta}{ds'} + 2\omega_\alpha \frac{d\xi^\alpha}{ds'} \frac{d\xi^\nu}{ds'} - \psi^{-2} g^{\nu\nu} \omega_\nu = 0 \tag{27}$$

und (3) gibt uns die Ausdrücke

$$\frac{d\xi^\nu}{ds'} = \psi^{-1} \frac{d\xi^\nu}{ds}\,, \quad \frac{d^2\xi^\nu}{ds'^2} = \psi^{-2} \frac{d^2\xi^\nu}{ds^2} - \psi^{-3} \psi_\mu \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds}\,. \tag{28}$$

Das Einsatzen (28) in (27) führt zu der Gleichung

$$\psi \frac{d^2\xi^\nu}{ds^2} + \psi \Gamma^\nu_{\alpha\beta} \frac{d\xi^\alpha}{ds} \frac{d\xi^\beta}{ds} + \psi_\alpha \frac{d\xi^\alpha}{ds} \frac{d\xi^\nu}{ds} - g^{\nu\nu} \psi_\nu = 0\,. \tag{29}$$

Mit der Bezeichnung

$$D = \frac{d\xi^\nu}{ds} \nabla_\nu \tag{30}$$

nimmt (29) die Form

$$D(\psi D\xi^\nu) = g^{\nu\nu} \nabla_\nu \psi \tag{31}$$

an. Dies sind die Bewegungsgleichungen im Minkowskischen Raume.

Im Inertialbezugssystem hat die Feldgleichung (26) die Gestalt

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \psi = \frac{\gamma}{4\pi} \int \delta_4(x^\mu - \xi^\mu)\,ds\,. \tag{32}$$

Bei Beachtung, dass in diesem Bezugssystem das Symbol (30) die gewöhnliche Differentiation bedeutet, bekommen wir die Bewegungsgleichungen

$$\frac{d}{ds}\left(\psi \frac{d\xi^\nu}{ds}\right) = \frac{\partial \psi}{\partial \xi^\nu}\,. \tag{33}$$

Die Lösung von (32) können wir immer in der Form

$$\psi = m + \varphi \tag{34}$$

darstellen, wo $m = \text{const}$ und φ auch eine Lösung von (32) ist. Nachdem wir (34) in (33) eingesetzt haben, erhalten wir

$$\frac{d}{ds}\left[(m+\varphi)\frac{d\xi^\nu}{ds}\right] = \frac{\partial \varphi}{\partial \xi^\nu}. \tag{35}$$

Diese Gleichung beschreibt die Bewegung eines Teilchens, dessen Eigenmasse vom Feld φ abhängt [2], [4].

FAKULTÄT FÜR THEORETISCHE PHYSIK, UNIWERSITÄT ZU WROCŁAW
MATHEMATISCHES INSTITUT (ABTEILUNG WROCŁAW), POLNISCHE AKADEMIE DER WISSENSCHAFTEN

SCHRIFTTUM

[1] G. Szamosi, Acta Phys. Hung., **6** (1956), 207.
[2] L. Infeld, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 491.
[3] J. A. Schouten und D. J. Struik, *Einführung in die neuere Methoden der Differentialgeometrie*, B. II, Groningen, 1938.
[4] J. Werle, Bull. Acad. Polon. Sci., Cl. III, **1** (1953), 281.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

Über die Bewegung der Feldsingularitäten in kovarianten Feldtheorien

von

C. JANKIEWICZ

Vorgelegt von L. INFELD am 2 Oktober 1958

Einleitung

In der Arbeit [1] wurde ein Variationsprinzip formuliert, mit dessen Hilfe für kontinuierlich verteilte Feldquellen kovariante Bewegungsgleichungen abgeleitet wurden. Ebenda wurde auch gezeigt, dass diese Bewegungsgleichungen aus den kovarianten Feldgleichungen folgen und dass sie mit den verallgemeinerten Erhaltungsgesetzen für das Feld identisch sind. In der vorliegenden Note wird gezeigt, dass beim Übergang mit Hilfe der Diracschen δ-Funktion zu unstetigen Quellen die oben erwähnten Gleichungen für die kontinuierlich verteilten Feldquellen in Bewegungsgleichungen der Feldsingularitäten übergehen. Auf diese Weise wird bewiesen, dass in einer jeden auf Grund des allgemeinen Relativitätsprinzipes formulierten klassischen Feldtheorie die Bewegungsgleichungen der Feldsingularitäten aus den Feldgleichungen folgen. Im Unterschied zu den Ergebnissen von Bergmann [2] und Goldberg [3] haben die von uns erhaltene Bewegungsgleichungen die Form Lagrange-Eulerscher Gleichungen eines entsprechenden Variations-prinzipes.

Bewegungsgleichungen für stetige Quellen

Im Einklang mit den Annahmen des allgemeinen Relativitätsprinzipes beschreiben wir die Feldquellen mit Hilfe eines linearen geometrischen Obiektes $\psi_A(x^\nu)$ und das Feld mit Hilfe M ebensolcher Obiekte $\psi_{IA}(x^\nu)$. Die Wirkungsfunktion nehmen wir für das betrachtete physikalische System in der Form

$$S = \int \Phi(\psi_A, \psi_{IA}, \partial_a \psi_{IA}, \ldots)(dx) \tag{1}$$

an, wo Φ eine geometrische Komitante ist, die eine Skalardichte mit dem Gewicht $+1$ ist.

Die Bewegungsgleichungen für stetig verteilte Quellen definieren wir durch [1]

$$D[\psi_A]S = 0\,, \tag{2}$$

wo $D[\psi_A]S$ ein Liesches Partialdifferential bezeichnet. Aus der Bedingung (2) folgen die Bewegungsgleichungen in der Form

$$\partial_\mu T^\mu_\nu - F_\nu = 0\,. \tag{3}$$

Die Obiekte T^μ_ν und F_ν hängen von der Komitante Φ und den Transformationsgesetzen des Obiektes ψ_A ab. Die entsprechenden Formeln werden wir im Laufe der Arbeit unmittelbar angeben.

Bewegungsgleichungen für Feldsingularitäten

Der Wirkungsfunktion für das Feld und seine Singularitäten geben wir die Form

$$S = \int \Big[\int \sum_{i=1}^{N} \delta(x-\xi_i)\, \Phi(\dot{\xi}^\mu_i,\ \psi_{IA}\partial_a\psi_{IA}\ \ldots)\, ds_i + \Phi'(\psi_{IA},\ \partial_a\psi_{IA},\ldots)\Big](dx)\,, \tag{4}$$

wo die Funktionen $\xi^\mu_i(s_i)$ die Bewegung der Singularitäten beschreiben und $\dot{\xi}^\mu_i = \dfrac{d\xi^\mu_i}{ds_i}$ und s_i beliebige Parameter sind. (Hier bedeutet $\delta(x-\xi_i)$ die von Infeld und Plebański modifizierte vierdimensionelle Diracsche $\hat{\delta}$-Funktion

$$\int \sum_i \delta(x-\xi_i)\, ds_i = \Big| \sum_i \hat{\delta}(\vec{x}-\vec{\xi}_i)\, \frac{ds_i}{dt_i}\Big|_{\tau_i = t}\ [5]\,.$$

Wie bekannt, können wir die Bewegungsgleichungen der Feldsingularitäten aus dem Variationsprinzip

$$\delta[\xi^\mu_i]S = 0 \tag{5}$$

erhalten. Aus (5) bekommen wir

$$\delta S = \int\int \sum_i \left[\frac{\partial\delta}{\partial\xi^\nu_i}\Phi\delta\xi^\nu_i + \delta\,\frac{\partial\Phi}{\partial\dot{\xi}^\mu_i}\delta\dot{\xi}^\mu_i \right] ds_i(dx)\,. \tag{6}$$

Die partielle Integration liefert

$$\delta S = \int\int \sum_i \left[\frac{\partial\delta}{\partial\xi^\nu_i}\Phi\delta\xi^\nu_i - \frac{d}{ds_i}\left(\delta\,\frac{\partial\Phi}{\partial\dot{\xi}^\nu_i}\right)\delta\xi^\nu_i \right] ds_i(dx)\,. \tag{7}$$

Bei Beachtung der Eigenschaften der δ-Funktion können wir (7) auf die Form

$$\delta S = \int\int \sum_i \delta(x-\xi_i)\left[-\dot{\xi}_i^\nu \frac{\partial}{\partial x^\nu}\frac{\partial \Phi}{\partial \dot{\xi}_i^\mu} - \ddot{\xi}_i^\nu \frac{\partial^2 \Phi}{\partial \dot{\xi}_i^\nu \partial \dot{\xi}_i^\mu} + \frac{\partial}{\partial x_\mu}\Phi\right]\delta\xi_i^\mu \, ds_i(dx) \tag{8}$$

bringen.

Wenn wir die Integration in voneinander getrennten Umgebungen der Punkte ξ_i^μ ausführen, bekommen wir die Bewegungsgleichungen in der Lagrange-Eulerschen Gestalt

$$\frac{\partial \tilde{\Phi}_i}{\partial \xi_i^\mu} - \frac{d}{ds_i}\frac{\partial \tilde{\Phi}_i}{\partial \dot{\xi}_i^\mu} = 0, \tag{9}$$

wo $\tilde{\Phi}_i = \Phi(\dot{\xi}_i^\mu, \psi_{IA}(\xi_i^\mu))$ eine reguläre Funktion ist [5].

Der Übergang zu unstetigen Quellen

Jetzt zeigen wir, dass die Gleichungen (3) für die Wirkungsfunktion (4) die Form (9) haben. In diesem Fall sind Bewegungsfunktionen: die Skalardichte $\delta(x-\xi_i)$ mit dem Gewicht $+1$ und die kontravarianten Vektoren $\dot{\xi}_i^\mu$. Für das Liesche Partialdifferential von (4) haben wir

$$DS = \int\int \sum_i \left[\Phi D\delta(x-\xi_i) + \delta\frac{\partial \Phi}{\partial \dot{\xi}_i^\mu} D\dot{\xi}_i^\mu\right] ds_i(dx). \tag{10}$$

Ähnlich geben die Formeln für das Liesche Differential einer Skalardichte und eines kontravarianten Vektors (s. z. B. [4])

$$D\delta(x-\xi_i) = \frac{\partial}{\partial x^\mu}[\delta(x-\xi_i)\omega^\mu]d\varepsilon \tag{11}$$

und

$$D\dot{\xi}_i^\mu = -\dot{\xi}_i^a \frac{\partial}{\partial x^a}\omega^\mu d\varepsilon, \tag{12}$$

wo ω^μ der Raumdeformationsvektor und $d\varepsilon$ eine infinitesimale Grösse ist. Das Einsetzen von (11) und (12) in (10) führt zu

$$DS = \int\int \sum_i \left[\Phi\frac{\partial}{\partial x^\mu}(\delta\omega^\mu) - \delta\frac{\partial \Phi}{\partial \dot{\xi}_i^\mu}\dot{\xi}_i^a \frac{\partial}{\partial x^a}\omega^\mu\right] d\varepsilon \, ds_i(dx) \tag{13}$$

und nach der partiellen Integration zu

$$DS = \int\int \sum_i \left[-\delta\frac{\partial \Phi}{\partial x^\mu} + \frac{\partial}{\partial x^a}\left(\delta\frac{\partial \Phi}{\partial \dot{\xi}_i^\mu}\dot{\xi}_i^a\right)\right]\omega^\mu d\varepsilon \, ds_i(dx), \tag{14}$$

woraus die Bewegungsgleichungen in der Form (3) folgen mit

$$T_\nu^\mu = \int \sum_i \delta(x-\xi_i)\frac{\partial \Phi}{\partial \dot{\xi}_i^\nu}\dot{\xi}_i^\mu \, ds_i \tag{15}$$

und

$$F_\nu = \int \sum_i \delta(x-\xi_i)\frac{\partial \Phi}{\partial x^\nu} ds_i \,. \tag{16}$$

Aus (3), (15) und (16) erhält man die Bewegungsgleichungen in einer "scheinbar feldlichen" Form

$$\int \sum_i \left[\frac{\partial}{\partial x^\mu}\left(\delta \frac{\partial \Phi}{\partial \dot{\xi}_i^\nu}\dot{\xi}_i^\mu\right) - \delta\frac{\partial \Phi}{\partial x^\nu}\right] ds_i \,; \tag{17}$$

man kann (17) weiter in

$$\int \sum_i \left[-\frac{\partial \delta}{\partial \xi_i^\mu}\frac{\partial \Phi}{\partial \dot{\xi}_i^\nu}\dot{\xi}_i^\mu + \delta\dot{\xi}_i^\mu \frac{\partial}{\partial x^\mu}\frac{\partial \Phi}{\partial \dot{\xi}_i^\nu} - \delta\frac{\partial \Phi}{\partial x^\nu}\right] ds_i = 0 \tag{18}$$

und, nach partieller Intergration, in

$$-\int \sum_i \delta\left(-\frac{\partial \Phi}{\partial x^\nu} + \dot{\xi}_i^\mu \frac{\partial}{\partial x^\mu}\frac{\partial \Phi}{\partial \dot{\xi}_i^\nu} + \ddot{\xi}_i^\mu \frac{\partial^2 \Phi}{\partial \dot{\xi}_i^\nu \partial \dot{\xi}_i^\mu}\right) ds_i = 0 \tag{19}$$

umformen.

Wenn wir die Integration in (19) in voneinander getrennten Umgebungen der ξ_i^μ ausführen, bekommen wir die Bewegungsgleichungen (9).

Bei Beachtung, das die Bewegungsgleichungen (3) aus den Feldgleichungen (1) folgen, kommen wir zu folgenden Ergebnissen:

a) Beim Übergang zu den als Feldsingularitäten aufgefassten Quellen, d. h. bei der Annahme der Wirkungsfunktion in der Form (4) sind die Gleichungen (9) identisch den Gleichungen (3).

b) In einer jeden kovarianten Feldtheorie, die durch die Wirkungsfunktion (4) charakterisiert wird, folgen die Bewegungsgleichungen (9) für die Feldsingularitäten aus den Feldgleichungen.

In der allgemeinen Relativitätstheorie stellt (9) die Gleichungen der „geodätischen Linien" dar. Dass sie aus den Gravitationsfeldgleichungen folgen, haben Infeld und Plebański [5] gezeigt.

Der Verfasser dieser Arbeit möchte hiermit Herrn W. Zientek für kritische Hinweise seinen Dank aussprechen.

FAKULTÄT FÜR THEORETISCHE PHYSIK AN DER UNIVERSITÄT IN WROCŁAW
MATHEMATISCHES INSTITUT (ABTEILUNG WROCŁAW), POLNISCHE AKADEMIE DER WISSENSCHAFTEN

SCHRIFTTUM

[1] Cz. Jankiewicz, Acta Phys. Pol. (im Druck).
[2] P. G. Bergmann, Phys. Rev. **75** (1949), 4.
[3] J. Goldberg, Phys. Rev. **89** (1953), 263.
[4] K. Yano, *The theory of Lie derivatives and its applications*, Groningen, 1955.
[5] L. Infeld J. Plebański, Bull. Acad. Pol. Sci., Cl. III, **4** (1956), 11.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Serie des sci. chim., géol.
et géogr.—Vol. VI, No. 12, 1958

Muon Depolarization in High Energy Muon-Electron Scattering

by

A. DELOFF and R. GAJEWSKI

Presented by L. INFELD on October 11, 1958

In this paper we calculate the change of polarization of high energy muons due to their collisions with electrons. The results obtained are then used to estimate the contribution of depolarization originating from muon-electron scattering to the total depolarization occurring when the muons pass a layer of matter.

We follow the formalism introduced by Bouchiat and Michel [1] (further referred to as B. M.) who calculated the change of electron polarization in electron-electron scattering. In our case the incident particle is a polarized muon, whereas the scatterer is an unpolarized electron. Since these two particles are distinguishable, one must take into account only one Feynman graph, instead of two, as it was the case in electron-electron scattering. We assume the scattering electrons to be at rest in the laboratory system, and completely unpolarized.

Let us consider first a single act of scattering. We label the initial states of muon and electron by a and b, respectively. The final states of muon and electron are labeled by c and d. Let pseudo-vector $\boldsymbol{s}_a$ denote the polarization of the incident muon (bold face italics denote four component vectors). $\boldsymbol{s}_a$ can be expressed in terms of three vectors ${}_a\boldsymbol{n}_i$ $(i = 1, 2, 3)$ defined as follows

$$\text{(1a)} \qquad {}_a\boldsymbol{n}_i \boldsymbol{p}_a = 0\,, \qquad {}_a\boldsymbol{n}_k {}_a\boldsymbol{n}_l = -\delta_{kl} \quad (k, l = 1, 2), \qquad {}_a\boldsymbol{n}_3 {}_a\boldsymbol{n}_3 = -1$$

$$\text{(1b)} \qquad \varepsilon^{\lambda\mu\nu\varrho} p_{a\lambda} ({}_a\boldsymbol{n}_1)_\mu ({}_a\boldsymbol{n}_2)_\nu ({}_a\boldsymbol{n}_3)_\varrho = -M\,,$$

where $\boldsymbol{p}_a$ and M are, respectively, four-momentum and mass of the incident meson. The polarization four-vector can now be written as

$$\text{(2)} \qquad \boldsymbol{s}_a = \sum_{i=1}^{3} {}_a\zeta_i {}_a\boldsymbol{n}_i\,, \qquad -1 \leqslant {}_a\zeta_i \leqslant 1\,.$$

With a given choice of the basic vectors ${}_a\boldsymbol{n}_i$ the three parameters ${}_a\zeta_i$ fully determine the polarization of the incident muon. The degree of polarization is defined as

$$P_a = (-\boldsymbol{s}_a \boldsymbol{s}_a)^{1/2} = ({}_a\zeta_1^2 + {}_a\zeta_2^2 + {}_a\zeta_3^2), \qquad 0 \leqslant P_a \leqslant 1 . \tag{3}$$

The polarization of the scattered muon can be described by a pseudo-vector

$$\boldsymbol{s}_c = \sum_{j=1}^{3} {}_c\zeta_j \, {}_c\boldsymbol{n}_j , \qquad -1 \leqslant {}_c\zeta_j \leqslant 1 , \tag{4}$$

where the vectors ${}_c\boldsymbol{n}_j$ satisfy relations (1) with a substitution $a \to c$.

Three parameters ${}_c\zeta_j$ determining the final polarization can be found from a formula derived by B. M.

$${}_c\zeta_j = \sum_i C_{ji} \, {}_a\zeta_i , \tag{5}$$

where

$$\begin{aligned} C_{ji} &= A_{ij}/D , \\ A_{ij} &= \operatorname{Tr} M(\tau_i \odot 1)\, M^{+}(\tau_j \odot 1) , \\ D &= \operatorname{Tr} M\, M^{+} , \end{aligned} \tag{6}$$

τ_i are 2×2 Pauli matrices, 1 is a unit 2×2 matrix, the sign $\odot$ denotes direct product; M is the transition matrix for muon-electron scattering expressed in a representation used by B. M. This matrix can be calculated by considering the corresponding second-order Feynman graph. After evaluating the traces and using (1a) we get

$$\begin{aligned} \tfrac{1}{2} A_{ij} = {} & ({}_a\boldsymbol{n}_i \, {}_c\boldsymbol{n}_j) N - m^2 ({}_c\boldsymbol{n}_j \boldsymbol{p}_a)({}_a\boldsymbol{n}_i \boldsymbol{p}_c) + 2(\boldsymbol{p}_a \boldsymbol{p}_b)({}_a\boldsymbol{n}_i \boldsymbol{p}_c)({}_c\boldsymbol{n}_j \boldsymbol{p}_b) \\ & + 2(\boldsymbol{p}_b \boldsymbol{p}_c)({}_c\boldsymbol{n}_j \boldsymbol{p}_a)({}_a\boldsymbol{n}_i \boldsymbol{p}_b) + 2(M^2 - \boldsymbol{p}_a \boldsymbol{p}_c)({}_a\boldsymbol{n}_j \boldsymbol{p}_b)({}_a\boldsymbol{n}_i \boldsymbol{p}_b) , \end{aligned} \tag{7}$$

$$\tfrac{1}{2} D = -N - (\boldsymbol{p}_a \boldsymbol{p}_b - \boldsymbol{p}_b \boldsymbol{p}_c)(M^2 - \boldsymbol{p}_a \boldsymbol{p}_c) , \tag{8}$$

where

$$N = -m^2 M^2 - 2(\boldsymbol{p}_a \boldsymbol{p}_b)(\boldsymbol{p}_c \boldsymbol{p}_b) + M^2(\boldsymbol{p}_a \boldsymbol{p}_b - \boldsymbol{p}_b \boldsymbol{p}_c) + m^2 \boldsymbol{p}_a \boldsymbol{p}_c , \tag{9}$$

and m denotes the mass of the electron.

We now chose the 6 vectors ${}_a\boldsymbol{n}_i$, ${}_c\boldsymbol{n}_j$ as follows

$$\begin{aligned} & {}_a\boldsymbol{n}_1(0, \vec{x}_1), \quad {}_a\boldsymbol{n}_2(0, \vec{x}_2), \quad {}_a\boldsymbol{n}_3\left(\frac{p_a}{M}, \frac{E_a}{M}\frac{\vec{p}_a}{p_a}\right), \\ & {}_c\boldsymbol{n}_1(0, \vec{x}_1'), \quad {}_c\boldsymbol{n}_2(0, \vec{x}_2), \quad {}_c\boldsymbol{n}_3\left(\frac{p_c}{M}, \frac{E_c}{M}\frac{\vec{p}_c}{p_c}\right), \end{aligned} \tag{10}$$

where $\vec{x}_1$ and $\vec{x}_2$ are unit vectors perpendicular to each other and perpendicular to $\vec{p}_a$, $\vec{x}_2$ being perpendicular to the plane of scattering; $\vec{x}_1'$ is a unit vector defined as $\vec{x}_1' = \vec{x}_2 \times \vec{p}_c / p_c$. E and $\vec{p}$ denote, respectively, energy and momentum mesured in the laboratory system, $p = |\vec{p}|$.

Having in mind that

$$p_a = (E_a, \vec{p}_a), \quad p_b = (m, 0), \quad p_c = (E_c, \vec{p}_c)$$

we have, with the above choice of vectors $\boldsymbol{n}$,

$$\begin{aligned}
(11)\quad \tfrac{1}{2}A_{11} &= -N\cos\Theta + m^2 p_a p_c \sin^2\Theta\,,\\
\tfrac{1}{2}A_{22} &= -N\,,\\
\tfrac{1}{2}A_{33} &= -\frac{N}{M^2}(p_a p_c - E_a E_c\cos\Theta)\\
&\quad -\frac{m^2}{M^2}[-E_a E_c p_a p_c \sin^2\Theta + M^2\cos\Theta(p_a^2+p_c^2) - 2M^2 p_a p_c]\,,\\
\tfrac{1}{2}A_{13} &= -\frac{\sin\Theta}{M}[E_c N + m^2 p_c(E_a p_c + E_c p_a\cos\Theta)]\,,\\
\tfrac{1}{2}A_{31} &= \frac{\sin\Theta}{M}[E_a N + m^2 p_a(E_c p_a + E_a p_c\cos\Theta)]\,,\\
\tfrac{1}{2}D &= -N - m(E_a - E_c)(M^2 - E_a E_c + p_a p_c\cos\Theta)\,,
\end{aligned}$$

where Θ is the scattering angle mesured in the laboratory system. Furthermore, we have, according to (9),

$$(12)\qquad N = -m^2M^2 + mM^2(E_a - E_c) - m^2(E_a E_c + p_a p_c\cos\Theta)\,.$$

Now we introduce the assumption that the energy transfer $q = E_a - E_c$ is much smaller than the kinetic energy of the meson $E - M$. Under this assumption we can develop all quantities entering (11) in terms of q, and leave only terms leading in q. We have

$$(13)\quad E_c = E - q, \quad p_c \approx p - qE/p, \quad \sin\Theta \approx (2mq)^{1/2}/p, \quad \cos\Theta \approx 1 - mq/p^2,$$

where we dropped the index a at E and p.

Substituting (13) into (11) and (12) we get, according to (6),

$$(14)\qquad \begin{cases} C_{11} = C_{33} = 1 - \varepsilon q\,,\\ C_{22} = 1\,,\\ C_{31} = -C_{13} = (2q\varepsilon)^{1/2}\,, \end{cases}$$

where

$$(15)\qquad \varepsilon = \frac{mM^2}{p^2E^2}\,.$$

Formula (5) can now be written as

$$(16)\qquad \begin{pmatrix} \zeta_1' \\ \zeta_2' \\ \zeta_3' \end{pmatrix} = \begin{pmatrix} 1-\varepsilon q & 0 & -(2q\varepsilon)^{1/2} \\ 0 & 1 & 0 \\ (2q\varepsilon)^{1/2} & 0 & 1-\varepsilon q \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix},$$

where $\zeta_1, \zeta_2, \zeta_3$ denote the components of the 'polarization vector' for the incident muon, whereas $\zeta_{1'}, \zeta_{2'}, \zeta_{3'}$ denote the components of the 'polarization vector' for the scattered muon. However, these components are taken with respect to two different co-ordinate systems: the system $\vec{x}_1, \vec{x}_2, \vec{x}_3$ to which $\vec{\zeta}$ is referred differs from the system $\vec{x}_1' \vec{x}_2' = \vec{x}_2, \vec{x}_3'$ to which $\vec{\zeta}'$ is referred, the latter being obtained from the former by a rotation about the axis x_2 through the angle Θ. Thus, co-ordinates $\zeta_1', \zeta_2', \zeta_3'$ of ζ' are obtained from the co-ordinates $\zeta_{1'}, \zeta_{2'}, \zeta_{3'}$ by rotating the co-ordinate system through the angle Θ. Now, since transformation (16) describes a rotation through a small angle φ

$$\varphi = (2q\varepsilon)^{1/2} \approx \sin\varphi\,, \tag{17}$$

we conclude that the net result of scattering is the rotation of $\vec{\zeta}$ about the axis x_2, through an angle $\Theta - \varphi$,

$$\begin{pmatrix} \zeta_1' \\ \zeta_2' \\ \zeta_3' \end{pmatrix} = \begin{pmatrix} \cos(\Theta-\varphi) & 0 & \sin(\Theta-\varphi) \\ 0 & 1 & 0 \\ -\sin(\Theta-\varphi) & 0 & \cos(\Theta-\varphi) \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}. \tag{18}$$

From (13) we have $\Theta = (2mq)^{1/2}/p$. Comparison with (17) yields

$$\varphi = \frac{M}{E}\Theta \tag{19}$$

from which it follows that $\varphi \leqslant \Theta$; thus, the change in the direction of the polarization never exceeds the change in the direction of the momentum. For small kinetic energies we have $\varphi \approx \Theta$, and there is no change in the direction of the polarization. In the case of high energies we have $\varphi \to 0$, and the direction of the polarization follows the direction of the momentum.

Formula (19) describing the change of muon polarization due to a single muon-electron scattering is exactly the same as in the case where the scatterer is a nucleus, rather than an electron. Since, in small angle approximation, the scattering cross-section in muon-electron scattering is Z times smaller than that in Coulomb scattering, we infer that the influence of electrons upon the depolarization of muons is just the same as would be the influence of nuclei with a scattering cross-section Z times smaller than the actual one, i. e. with atomic charge number $\sqrt{Z}$. The problem is thus reduced to the problem of muon depolarization due to multiple Coulomb scattering, which was investigated by Hayakawa [2] *).

*) A similar problem of electron depolarization due to multiple Coulomb scattering has been investigated in greater detail by Mühlschlegel and Koppe [3].

We infer that the contribution to muon depolarization arising from collisions with atomic electrons is Z times smaller than the depolarization caused by Coulomb scattering.

We are indebted to Professor J. Werle for suggesting to us the problem.

INSTITUTE FOR NUCLEAR RESEARCH, WARSAW

REFERENCES

[1] C. Bouchiat and L. Michel, Nucl. Phys. **5** (1958), 416.
[2] S. Hayakawa, Phys. Rev. **108** (1957), 1533.
[3] B. Mühlschlegel and H. Koppe, Zts. f. Phys. **150** (1958), 474.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 12, 1958

EXPERIMENTAL PHYSICS

Fluorescence of Solutions of Eosine in Glucose

by

D. FRĄCKOWIAK, L. KORBUTT and T. KORBUTT

Presented by A. JABLONSKI on October 7, 1958

This work was undertaken with the aim of providing an answer to the question whether, in solutions of eosine in glucose, complexes of dye molecules are formed, as well as complexes of the same molecules with glucose. An earlier investigation [1] proved complexes of both kinds to form in solutions of fluorescein in glucose. The solutions investigated were prepared as previously [1], [2].

Absorption and emission spectra, and the fluorescence yield of solutions of 2×10^{-5} g of the dye per 1 g glucose, prepared by boiling at various temperatures under various pressures were investigated. The fluorescence band exhibiting a maximum about 5500 Å undergoes neither deformation nor displacement as the ebullition temperature varies, and is probably emitted by the non-associated molecules of the dye, whereas the shape of the absorption band varies considerably as the ebullition temperature rises. It is clear from Fig. 1 that, as the temperature rises, the intensity of the absorption maximum at about 5300 Å diminishes with respect to that of the maximum in the vicinity of 5100 Å. As temperature rises, the band narrows down at the shortwave side.

The fluorescence yield was measured by a method already described [3]. Fig. 2 shows the absolute fluorescence yields of solutions of eosine in glucose versus the wave-length of the exciting light. Measurement yielded directly the ratio of the fluorescence yield of the eosine solution in glucose to that of an aqueous solution of eosine, for a given exciting wave-length. For wave-lengths shorter than 5470 Å, the results of measurements carried out by S. I. Vavilov [4] on aqueous eosine solutions were used for conversion of the relative fluorescence yields measured into absolute fluorescence yields. For longer wave-lengths, the fluorescence yields of aqueous solutions were measured by the present authors. It will be seen from Fig. 2 that, as the temperature of ebullition rises, the absolute fluorescence yield slightly increases, and the slope of the yield beginning at about 5450 Å becomes less steep.

The results obtained are satisfactorily accounted for by the earlier hypothesis [2] which assumes that, in highly viscous solutions in glucose, the molecules, on cooling, remain in the state of association reached during ebullition of the solution. The absorption maximum at about 5100 Å corresponds to eosine monomers, whilst that at about 5300 Å,

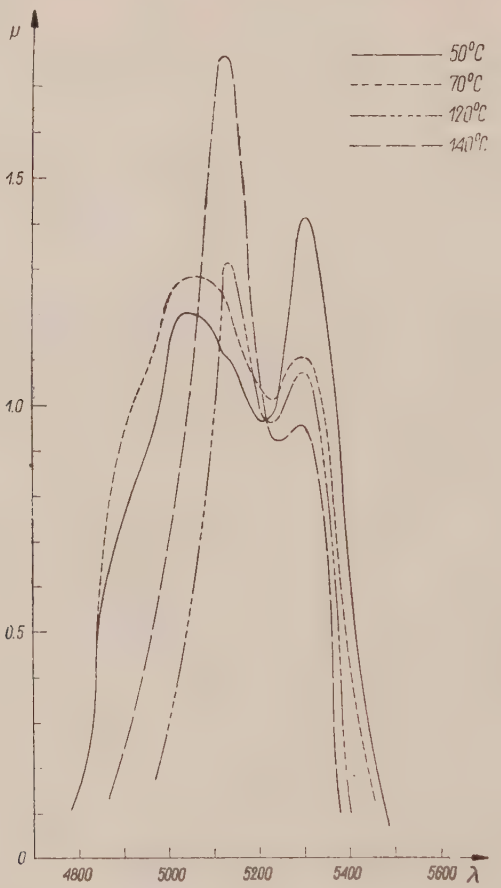

Fig. 1. Absorption spectrum of eosine solution in glucose

———— temperature of ebullition of solution 50°C
- - - - - - - - - - „ „ „ „ „ 70°C
— - - - - — „ „ „ „ „ 120°C
—— — —— „ „ „ „ „ 140°C

whose intensity diminishes with respect to the intensity of the monomer band as the ebullition temperature rises, corresponds to dimers of eosine. Foerster [5] showed as many as two maxima to exist in the absorption band corresponding to dimers in aqueous solutions of eosine (Table I). The shrinking of the short-wave end of the absorption band (Fig. 1) is possibly due to the diminishing, with rising ebullition temperatures of

the solution, of a short-wave dimer band not resolved from that of the monomers.

It may be seen from Table I that the maxima in the absorption bands of the eosine monomers and dimers in glucose solutions are displaced with respect to those in aqueous solutions. It seems probable that eosine does not form complexes with glucose.

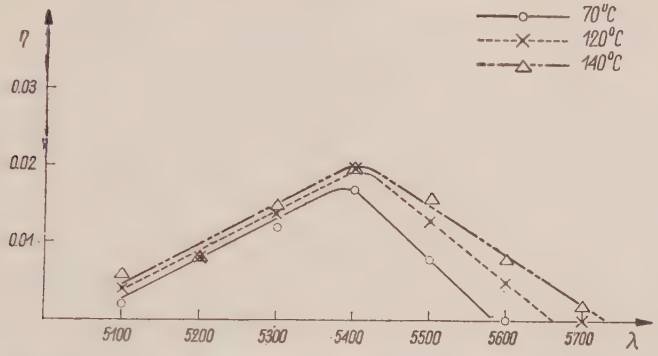

Fig. 2. Absolute fluorescence yield of eosine solution in glucose versus wave-length of exciting light for various temperatures of ebullition of solution

TABLE I

Absorption spectrum of eosine

	I. dimer band	monomer band	II. dimer band
eosine in aqueous solution (Foerster)	4950 Å	5170 Å	5220 Å
eosine in glucose solution	5000 Å	5130 Å	5300 Å

The fact that the fluorescence yield is higher and its drop in the anti-Stokes region is less rapid for specimens prepared at higher boiling temperatures is satisfactorily explained by the hypothesis of Foerster [7] and Jabłoński [6].

According to this hypothesis the drop of yield in the anti-Stokes region of excitation is due to absorption by non-luminescent dimers, whose concentration is lower in specimens prepared at higher temperatures.

DEPARTMENT OF PHYSICS, NICHOLAS COPERNICUS UNIVERSITY, TORUŃ

REFERENCES

[1] D. Frąckowiak, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 991.
[2] R. Drabent, D. Frąckowiak, Acta Phys. Polon. **14** (1955), 447.
[3] R. Bauer, D. Frąckowiak, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 729.
[4] S. I. Vavilov, Phil. Mag. **43** (1922), 307.
[5] Th. Foerster, E. Koenig, Zts. f. Elektrochemie, **61** (1957), 344.
[6] A. Jabłoński, Acta Phys. Polon., **13** (1954), 239.
[7] Th. Foerster, *Fluoreszenz organischer Verbindungen*, Goettingen, 1951, p. 152.

TABLE DES MATIÈRES DU VOLUME VI

Mathématique

R. M. Friedberg, 4-Quantifier Completeness: A Banach-Mazur Functional not Uniformly Partial Recursive 1

S. Mrówka, On the Potency of Compact Spaces and the First Axiom of Countability 7

A. Pliś, On Contact Points of Two Integrals of a Partial Differential Equation of the First Order 11

A. Bielecki et J. Kisyński, Une remarque à propos de deux notes de Z. Szmydt 15

M. Altman, On the Approximate Solutions of Non-Linear Functional Equations in Banach Spaces 19

А. Гранас, О непрерывных отображениях открытых множеств в банаховых пространствах 25

Z. Szmydt, Sur l'existence de solution de certains problèmes aux limites relatifs à un système d'équations différentielles hyperboliques 31

A. Hulanicki, On Cardinal Numbers Related with Locally Compact Groups 67

A. Hulanicki, Algebraic Structure of Compact Abelian Groups 71

W. Oktaba, On the linear Hypothesis in the Theory of Normal Regression . 75

W. Pogorzelski, Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles 79

W. Pogorzelski, Propriété d'une intégrale singulière pour les arcs non fermés et son application 85

S. Mrówka, A Generalization of a Theorem Concerning the Power of a Perfect Compact Metric Space 89

S. Mrówka, A Property of Hewitt Extension rX of Topological Spaces . . . 95

C. Bessaga, A Note on Universal Banach Spaces of a Finite Dimension . . . 97

T. Traczyk, On Homomorphisms not Induced by Mappings 103

H. Milicer-Grużewska, Un théorème limite sur la dérivée de l'intégrale de Poisson-Weierstrass généralisée 131

A. Alexiewicz and Z. Semadeni, A Generalization of Two Norm Spaces. Linear Functionals 135

Ou Ŝo-Mo, Note on Some Function-Theoretic Inequalities 141

A. Pliś and T. Ważewski, A Uniqueness Condition with a Standard Differential Equation without Uniqueness Property 145

K. Maurin, Entwicklung positiv definiter Kerne nach Eigendistributionen. Differenzierbarkeit der Spektralfunktion eines hypoelliptischen Operators . . . 149

J. Krzyż, On the Derivative of Bounded Univalent Functions 157

S. Mrówka, An Example of a Non-normal Completely Regular Space 161

A. Schinzel, Sur un problème concernant le nombre de diviseurs d'un nombre naturel 165

J. Mioduszewski, Solution générale d'un problème de Sikorski 169

A. Rybarski, Über eine gewisse Linearisationsmethode der Differentialgleichungen vom Pendeltypus 175

H. Heilbronn, On the Representation of Homotopic Classes by Regular Functions 181
L. Łukaszewicz, Accumulation of Errors in Approximate Calculation 233
M. Altman, On Extension of Linear Transformations in Banach Spaces . . . 241
C. Bessaga and A. Pełczyński, Some Remarks on Conjugate Spaces Containing Subspaces Isomorphic to the Space c_0 249
A. Pełczyński, A Connection Between Weakly Unconditional Convergence and Weakly Completeness of Banach Spaces 251
W. Janowski, Sur les valeurs extrémales du module de la dérivée des fonctions univalentes bornées. 255
S. Glidman, Some Theorems Concerning Determinants 275
K. Kuratowski, Sur l'extension de la notion de fonction rationnelle à l'espace euclidien n-dimensionnel 281
Z. Semadeni, A Localization Theorem for Multiplicative Linear Functionals 289
M. Altman, An Extension to Locally Convex Spaces of Borsuk's Theorem on Antipodes 293
M. Altman, Continuous Transformations of Open Sets in Locally Convex Spaces 297
H. Steinhaus, On Shortest Paths on Closed Surfaces 303
S. Mrówka, A Remark Concerning the Multiplicative Linear Functionals. . . 309
C. Bessaga and A. Pełczyński, On Subspaces of a Space with an Absolute Basis 313
B. Piłat, Sur les extrêmes des fonctions composées par des intégrales des équations aux dérivées partielles du type elliptique 317
A. Bielecki et J. Kisyński, Sur un problème de Mlle Z. Szmydt relatif à l'équation $\partial^2 z/\partial z \partial y = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$ 321
W. Zawadowski, Axiomatic Characterization of some Rings of Real Functions 355
K. Sieklucki, On a Contractible Polytope which Cannot be Metrized in the Strong Convex Manner 361
S. Mrówka, On the Unions of Q-Spaces 365
W. Pogorzelski, Propriétés des solutions du système parabolique d'équations aux dérivées partielles 369
Z. Łuszczki, Sur le plongement k-indépendant d'espaces euclidiens 375
A. Kirkor, A Remark About Cartesian Division by a Segment 379
W. Słowikowski, On the Theory of Operator-Systems. II 383
L. Dubikajtis, On the Incidence Axioms of Various Geometries 423
R. Engelking and S. Mrówka, On E-Compact Spaces. 429
M. Fisz, A Central Limit Theorem for some Stochastic Processes 437
A. Špaček, Sur une caractérisation algébrique des espaces métriques. 445
J. Słomiński, Theory of Models with Infinitary Operations and Relations . . 449
Z. Semadeni, Sur les fonctionnelles linéaires dans des espaces vectoriels semi-ordonnés avec application à représentation des fonctionnelles par des intégrales 457
M. Fisz, A Limit Theorem for Non-Decreasing Random Functions. 485
A. Jankowski, The System of Axioms of the Möbius Geometry 489
H. Milicer-Grużewska, Propriété limite du potentiel généralisé de simple couche relativement à l'équation parabolique normale 495
Z. Ciesielski, On Absolute Convergence of Fourier Series of Almost All Functions of Wiener Space 501
A. Piskorek, Sur certains problèmes aux limites pour l'équation semi-linéaire parabolique normale 505
A. Lelek, Sur les involutions multivalentes 511

S. Jankowska, Les solutions du système d'équations $\varphi(x) \cdot \varphi(y)$ et $\sigma(x) \cdot \sigma(y)$ pour $x < y < 10000$ 541
P. Erdös, Solution of Two Problems of Jankowska 545
W. Orlicz, On the Summability of Bounded Sequences by Continuous Methods
J. Krzyż, A Symmetrization Result for Maximum Modulus 549
P. Szeptycki, A Simple Proof of an Imbedding Theorem of the Kondrashev Type 561
K. Kuratowski, Sur les composantes de l'éspace des transformations d'un espace localement compact en rétracte de voisinage 565
C. Olech, On the Characteristic Exponents of the Second Order Linear Ordinary Differential Equation. 573
W. Sieklucki, Topological Properties of Sets Admitting the Tschebycheff System 603
L. Dubikajtis, On the Order of Points and Hyperplanes in n-dimensional Projective Geometry. 607
Д. В. Гнеденко, О критерии Вилькоксона сравнения двух выборок. . . . 611
Z. Mikołajska et T. Ważewski, Sur les opérations transformant séries convergentes en séries convergentes 615
W. Mlak, Limitations and Dependence on Parameter of Solutions of Non-Stationary Almost Linear Differential Operators Equations 677
J. Kisyński, Sur la méthode des approximations successives pour un système de n équations à n inconnues 683
T. Leżański, Formula for the Spectral Decomposition 689
J. Mikusiński, Sur la fonction dont transformée de Laplace est $e^{-s^{\alpha}}$ 691
A. Pełczyński, On the Isomorphism of the Spaces m and M. 695
D. Przeworska-Rolewicz, Étude des systèmes à èquations intégrales singulières pour les arcs non fermés par la méthode des approximations successives 697
D. Przeworska-Rolewicz, Remarque sur un résultat de Kropatchev 727
E. Marczewski, A General Scheme of the Notions of Independence in Mathematics 731
J. Mikusiński, Sur les notions de distribution et d'opérateur 737
P. Erdös and K. Urbanik, On Sets Which Are Measured by Multiples of Irrational Numbers 743
S. Świerczkowski, On Independent Elements in Finitely Generated Algebras 749
K. Kuratowski, Cohomotopic Multiplication and Duality Theorems Concerning Arbitrary Subsets of Euclidean Space 753

Astronomie

W. Zonn and J. Stodółkiewicz, Modification of Fundamental Equation of Stellar Statistics in the Light of Discontinuity of Interstellar Matter . . . 185
W. Iwanowska, Subsystems, Populations and Masses of Stars 581

Physique Théorique

J. Sawicki, Deuteron Stripping on Spheroidal Nuclei I. 37
J. Sawicki, Deuteron Stripping on Spheroidal Nuclei II 45
J. Werle, Kinematic Depolarization of μ Mesons Emitted in π-Meson Decays 51
A. Deloff, The Longitudinal Polarization of Electrons in Once Forbidden β Transitions 57
J. Werle and J. Wrzecionko, Čerenkov Radiation of Polarized Electrons . . 191
M. Suffczyński, Three-Centre Integrals in Iron 195
D. Chlebowska, The (p, a) Reaction on Deformed Nuclei 199

R. Żelazny, On the Derivation of Hydrodynamical Equations for the Classical Systems of Diatomic Molecules 203
J. Rzewuski, The Spinor Space 261
A. Deloff, The Coulomb Effect in β Decay 327
J. Rzewuski, Two Theorems Concerning the Field Equations in the Spinor Space 335
Z. Szymański, Excited Levels of Nuclei with Mixed Configurations. I. Method of Calculation 343
J. Wrzecionko, Beta-Decays and Time Reversal 387
B. Kuchowicz, Tests for Time Reversal Invariance and Feynman-Gell-Mann Theory in Allowed Beta Transitions 395
A. Trautman, Boundary Conditions at Infinity for Physical Theories 403
A. Trautman, Radiation and Boundary Conditions in the Theory of Gravitation 407
W. Królikowski, On the Isobars of the Nucleon in the Fixed-Source Theory 413
W. Królikowski, On the Weak Interaction of Elementary Particles. 417
R. Rączka and A. Rączka, The Möller Scattering of Arbitrarily Polarized Electrons 463
Z. Szymański, Excited Levels of Nuclei with Mixted Configurations. II. Numerical Example 469
J. Werle, Transition Matrices for Reactions between Two Bosons and Two Fermions 477
M. Suffczyński, On some Applications of the OPW Calculations 481
H. Cofta, Spin Waves Frequency Spectrum for Various Orders of Ferromagnetic Translation Lattices 519
W. Królikowski, On the Classification of Elementary Fermions 523
A. Deloff and R. Gajewski, Angular Distribution of High-Energy Bremsstrahlung 529
S. Świerczkowski and K. F. Wojciechowski, On the Order-Disorder Transition Temperature in n-Component Alloys. 585
S. G. Rohoziński, Electron-Positron Pair Annihilation at Arbitrary Polarizations 619
A. Trautman, On Gravitational Radiation Damping 627
J. Dąbrowski, Binding Energy of the ^{16}O Nucleus with the Cluster Development Method 635
W. Tulczyjew, On the Motion of Rotating Bodies in the General Theory of Relativity 645
A. Pawlikowski and W. Szczurówna, On the Canonical Formalism for Dynamical Systems with Constrains 759
C. Jankiewicz, Über den Zusammenhang der vom Feld abhängigen Eigenmasse mit dem konform-Minkowskischen Raume 765
C. Jankiewicz, Über die Bewegung der Feldsingularitäten in kovarianten Feldtheorien 771
A. Deloff and R. Gajewski, Muon Depolarization in High Energy Muon-Electron Scattering 775

Physique Expérimentale

A. Pilawski, Investigation of Ferroelectrics by Temperature Measurements 61
M. Czajkowski and J. Grzywacz, Dependence of Polarization of Fluorescence on the Concentration of Luminescent Molecules in Plexiglassphosphors . . 107
R. Bauer and A. Bączyński, On the Exciting Wave Length Dependence of the Ratio of the Yields of Phosphorescence and Fluorescence 113
A. Bączyński, W. Berdowski and S. Lęgowski, Solution of the Schrödinger Equation for the Van der Waals Potential 119

S. Łęgowski, Interaction Potential between a Mercury Atom and an Argon Atom 127
S. Kielich, Molecular Interaction in the Classical Theory of Light Scattering 215
W. Szymanowski, R. Wiśniewski, R. Gołębiowski and I. Olszewska, Application of Pressed Powder Methods to Investigations of Infrared Absorption Spectra of Solids 223
A. Bączyński and M. Czajkowski, Polarization of Photoluminescence of Organophosphors 271
A. Kawski, On the Effect of Concentration on the Polarization of the Fluorescence of Rigid Solutions 533
A. Jabłoński, The Metastable State of Dye Molecules 589
H. Łożykowski and H. Męczyńska, Electroluminescence of CdS-Ag with the Ag_2S Layer 595
J. Stankowski, Non-Linear Effects in Rochelle Salt 599
R. Bauer, A. Bączyński and M. Czajkowski, Excitation Energy Transfer from Dye Molecules in the Metastable State 653
A. Tramer, Raman Spectrum of Antimony Trichloride Crystals 659
A. Jabłoński, Quenching of Photoluminescence of Solutions by Energy Transfer 663
A. Kawski, On the Dependence of Emission Anisotropy of Fluorescence on the Concentration of Luminescent Molecules in Plexiglass Luminophors 671
J. Grzywacz, The Influence of Torsional Vibrations of Luminescent Molecules on the Fundamental Polarization of Fluorescence 705
C. Bojarski, On the Temperature Dependence of the Fundamental Polarisation 713
C. Bojarski, Note on the Theory of the Self-Depolarisation of Photoluminescence of Solutions 719
D. Frąckowiak, L. Korbutt and T. Korbutt, Fluorescence of Solution of Eosine in Glucose 781

Géophysique

Z. Fajklewicz, On a Cracovian Method of Determining the Regional Gravity Field 349

LISTE DES AUTEURS DU VOLUME VI

Aleksiewicz A. 135
Altman M. 19, 241, 293, 297
Bauer R. 113, 653
Bączyński A. 113, 119, 271, 653
Berdowski W. 119
Bessaga C. 97, 249, 313
Bielecki A. 15, 321
Bojarski C. 713, 719
Chlebowska D. 199
Ciesielski Z. 501
Cofta H. 519
Czajkowski M. 107, 271, 653
Dąbrowski J. 635
Deloff A. 57, 327, 529, 775
Dubikajtis L. 423, 607
Engelking R. 429
Erdös P. 545, 743
Fajklewicz Z. 349
Fisz M. 437, 485
Frąckowiak D. 481, 781
Friedberg R. M. 1
Gajewski R. 529, 775
Glidman S. 275
Гнеденко Д. В. 611
Gołębiowski R. 223
Granas A. 25
Grzywacz J. 107, 705
Heilbronn H. 181
Hulanicki A. 67, 71
Iwanowska W. 581
Jabłoński A. 589, 663
Jankiewicz C. 765, 771
Jankowska S. 541
Jankowski A. 489
Janowski W. 255
Kawski A. 533, 671
Kielich S. 215
Kisyński J. 15, 321, 683
Kirkor A. 379
Korbutt L. 781
Korbutt T. 781
Królikowski W. 413, 417, 523
Krzyż J. 157
Kuchowicz B. 395
Kuratowski K. 281, 565, 753
Lelek A. 511
Leżański T. 689
Łęgowski S. 119, 127
Łożykowski J. 595
Łukaszewicz L. 233
Łuszczki Z. 375
Marczewski E. 731
Maurin K. 149
Męczyńska H. 595
Mikołajska Z. 615
Mikusiński J. 691, 737
Milicer-Grużewska H. 131, 495
Mioduszewski J. 169
Mlak W. 677
Mrówka S. . 7, 89, 95, 161, 309, 365, 429
Oktaba W. 75
Olech C. 573
Olszewska L. 223
Orlicz W. 549
Ou Ŝo-Mo 141
Pawlikowski A. 759
Pełczyński A. 249, 251, 313, 695
Pilawski A. 61
Piłat B. 317
Piskorek A. 505
Pliś A. 11, 145
Pogorzelski W. 79, 85, 369
Przeworska-Rolewicz D. . . . 697, 727
Rączka A. 463
Rączka R. 463
Rohoziński S. G. 619
Rybarski A. 175
Rzewuski J. 261, 335
Sawicki J. 37, 45
Schinzel A. 165
Semadeni Z. 135, 289, 457
Sieklucki K. 361, 603
Słomiński J. 449
Słowikowski W. 383

Špaček A. 445
Stankowski J. 599
Steinhaus H. 303
Stodółkiewicz J. 185
Suffczyński M. 195, 481
Szczurówna W. 759
Szeptycki P. 561
Szmydt Z. 31
Szymanowski W. 223
Szymański Z. 343, 469
Świerczkowski S. 585, 749
Traczyk T. 103
Tramer A. 659
Trautman A. 403, 407, 627
Tulczyjew W. 645
Urbanik K. 743
Ważewski T. 145, 615
Werle J. 51, 191, 477
Wiśniewski R. 223
Wojciechowski K. F. 585
Wrzecionko J. 191, 387
Zawadowski W. 355
Zonn W. 185
Żelazny R. 203

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 12

Д. ПШЕВОРСКАЯ-РОЛЕВИЧ, ЗАМЕТКА О НЕКОТОРОМ РЕЗУЛЬТАТЕ КРОПАЧЕВА . . . . . стр. 727—729

Дается нелинейное интегральное сингулярное уравнение

$$u(t) = \lambda \int_L \frac{K[t,\tau,u(\tau)]\,d\tau}{\tau - t}, \tag{1}$$

где L обозначает множество p замкнутых и непересекающихся линий на плоскости комплексной переменной, которые имеют непрерывную касательную в каждой точке.

В. Ф. Кропачев [1] приводит теорему, что если функция $K(t,\tau,u)$ удовлетворяет следующему условию

$$|K(t,\tau,u) - K(t',\tau',u')| \leqslant k[|t-t'|^{\nu} + |\tau-\tau'|^{\mu} + |u-u'|] \qquad (0 < \mu + \nu \leqslant 1), \tag{2}$$

то уравнение (1) имеет точно одно решение, полученное методом последовательных приближений, в пространстве всех функций, удовлетворяющих условию Гельдера со степенью μ при следующей норме

$$\|u\| = \sup_{t \in L} |u(t)| + \sup_{t,t' \in L} \frac{|u(t) - u(t')|}{|t-t'|^{\mu}}.$$

В настоящей работе приводится контр-пример к тезису Кропачева, а именно пример такой функции K, которая удовлетворяет условию (2), а операция, находящаяся с правой стороны уравнения (1), не удовлетворяет условию Липшица по отношению к норме.

Э. МАРЧЕВСКИЙ, ОБЩАЯ СХЕМА ПОНЯТИЯ НЕЗАВИСИМОСТИ В МАТЕМАТИКЕ . . . . . стр. 731—736

Рассматривается алгебра $(A, \boldsymbol{F})$, состоящая из некоторого множества A и множества $\boldsymbol{F}$ основных операций определенных в A. Через $\boldsymbol{A}^{(n)}$ обозначается множество алгебраических операций, зависящих от n переменных, т. е. функций n переменных являющихся суперпозициями основных операций.

Множество N, содержащееся в A, является множеством *независимых* элементов если для произвольной системы различных элементов $a_1, \ldots, a_n \in N$ и любой пары $g, h \in A^{(n)}$ из равенства $g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n)$ следует, что g и h тождественны в A.

Как частные случаи этого понятия получаются классические понятия: арифметическая независимость чисел, алгебраическая независимость чисел, линейная независимость векторов, независимость в смысле теории множеств, логическая независимость и алгебраическая независимость полиномов. Понятие независимости в теории вероятностей другого характера не содержится в рассматриваемой схеме.

ЯН МИКУСИНСКИЙ, О ПОНЯТИИ ДИСТРИБУЦИИ И ОБ ОПЕРАТОРЕ . . . . . стр. 737—741

Дистрибуция является оператором, если существует последовательность непрерывных функций, сходящаяся дистрибуционно к дистрибуции и операторно к оператору. Такое определение подтверждается фактом, что пространство непрерывных функций густо в пространстве дистрибуций и густо в пространстве операторов.

Чтобы дистрибуция была оператором, необходимо и достаточно, чтобы ее супорт был ограничен снизу.

Значение $e^{-s^{\alpha}\lambda}$ $(0 < \alpha < 1)$ является

1^0 функцией бесконечно дифференцируемой переменной t для значений λ, принадлежащих углу $|\arg\lambda| < \frac{\pi(1-\alpha)}{2}$ $(|\lambda| > 0)$;

2^0 дистрибуцией, которая не является функцией для значений λ. лежащих на полупрямых $|\arg\lambda| = \frac{\pi(1-\alpha)}{2}$;

3^0 оператором, который не является дистрибуцией для остальных значений λ.

П. ЭРДЕШ и К. УРБАНИК, О МНОЖЕСТВАХ ИЗМЕРЕННЫХ КРАТНЫМИ ИРРАЦИОНАЛЬНЫХ ЧИСЕЛ . . . . . стр. 743—748

Рассматривается класс Ξ всех таких множеств A $(A \subset [0, 1))$, что для произвольного иррационального числа ξ существует частота $\mathrm{fr}\{n : n\xi \in A (\mathrm{mod}\, 1)\}$, независящая от ξ.

Класс Ξ_0 называется базой (соответственно слабой базой) класса Ξ, если для любого $A \in \Xi$ существует $B \in \Xi_0$, такое, что

$$\mathrm{fr}\{n : n\xi \in A - B (\mathrm{mod}\, 1)\} = 0$$

для всех иррациональных ξ (соответственно для почти всех ξ).

В работе доказаны две теоремы:

Теорема 1. *Всякая база класса Ξ содержит $2^{2^{\aleph_0}}$ множеств неизмеримых по Лебегу.*

Теорема 2. *Всякая слабая база класса Ξ содержит не менее $2^{\aleph_1}$ множеств неизмеримых по Лебегу.*

С. СВЕРЧКОВСКИЙ, О НЕЗАВИСИМЫХ ЭЛЕМЕНТАХ В АЛГЕБРАХ С КОНЕЧНЫМ ЧИСЛОМ ОБРАЗУЮЩИХ . . . . . стр. 749—752

В работе доказаны три теоремы о независимых элементах (в смысле данном в работе Э. Марчевского) в алгебрах с конечным числом образующих.

Теорема 1. *Пусть A — алгебра с n-образующими, содержащая n независимых элетентов. Тогда любая система, состоящая из n-элементов, является системой образующих алгебры A тогда и только тогда, если они независимы.*

Теорема 2. *Пусть A — алгебра порожденная n-независимыми элементами и содержащая $n+1$ независимых элементов. Тогда для любого $m \geqslant n$ существует m независимых образующих алгебры A.*

Теорема 3. *Алгебра с n образующими, содержащая $n+1$ независимых элементов содержит бесконечное множество независимых элементов.*

К. КУРАТОВСКИЙ, КОГОМОТОПИЧЕСКОЕ УМНОЖЕНИЕ И ТЕОРЕМЫ ДВОЙСТВЕННОСТИ, КАСАЮЩИЕСЯ ПРОИЗВОЛЬНЫХ ПОДМНОЖЕСТВ ЕВКЛИДОВА ПРОСТРАНСТВА . . . . . стр. 753—758

Рассматривается с одной стороны топологическая группа $\mathfrak{L}(\mathcal{P}_n^{\mathcal{X}})$ всех компонент функционального пространства $\mathcal{P}_n^{\mathcal{X}}$ (где $\mathcal{X} \subset \mathcal{E}^n$, а $\mathcal{P}_n = \mathcal{E}^n - (0)$), с другой стороны — топологическая группа $\mathfrak{N}(\mathcal{S}_n - \mathcal{X})$ всех целочисленных мер (т. е. подчиненных условиям 1^0 и 2^0, § 4), определенных на множествах открыто-замкнутых по отношению к $\mathcal{S}_n - \mathcal{X}$.

А. ПАВЛИКОВСКИЙ и В. ЩУР, О КАНОНИЧЕСКОМ ФОРМАЛИЗМЕ ДЛЯ ДИНАМИЧЕСКИХ СИСТЕМ СО СВЯЗЯМИ . . . . . стр. 759—763

В работе сформулирован метод перехода к каноническому формализму для систем со связями определенного типа, которых Лагранжианы не содержат производных некоторых координат. Для этих случаев применение обычного метода невозможно. Настоящий метод позволяет сформулировать классическую аналогию теории коллективных переменных Бома-Пайнса и Мигдала-Галицкого.

Ч. ЯНКЕВИЧ, О СВЯЗИ СОБСТВЕННОЙ МАССЫ, ЗАВИСЯЩЕЙ ОТ ПОЛЯ С КОНФОРМНО-МИНКОВСКОВЫМ ПРОСТРАНСТВОМ стр. 765—769

Показано, что уравнения движения собственной массы, зависящей от скалярного ψ-поля, являются геодезическими конформно-Минковского пространства. Из уравнений Эйнштейна получено уравнение Даламбера для ψ-поля вне масс.

Ч. ЯНКЕВИЧ, О ДВИЖЕНИИ СИНГУЛЯРНОСТЕЙ ПОЛЯ В КОВАРИАНТНЫХ ТЕОРИЯХ . . . . . стр. 771—774

Показано, что в ковариантных теориях поля, охарактеризованных Лагранжианами линейными в δ-функции уравнения движения в форме Лагранжа-Эйлера следуют из уравнений поля.

А. ДЕЛОФФ и Р. ГАЕВСКИЙ, ДЕПОЛЯРИЗАЦИЯ МЕЗОНОВ μ ПРИ РАССЕИВАНИИ ВЫСОКО ЭНЕРГЕТИЧЕСКИХ МЕЗОНОВ ЭЛЕКТРОНАМИ . . . . . стр. 775—779

Исследована деполяризация поляризованных мезонов μ вследствие столкновений с покоящимися электронами.

Показано, что при столкновениях быстрых мезонов μ с атомами — деполяризация происходящая при столкновении с электронами в Z раз меньше, чем деполяризация, происходящая при столкновении с ядром.

Д. ФРОНЦКОВЯК, Л. КОРБУТТ, Т. КОРБУТТ, ИССЛЕДОВАНИЕ ФЛУОРЕСЦЕНЦИИ РАСТВОРА ЭОЗИНА В ГЛЮКОЗЕ . . стр. 781—783

Исследованы абсорбционные и эмиссионные спектры, а также зависимость выхода флуоресценции от длины волны возбуждающего света для приготовленных в разных температурах растворов эозина в глюкозе. На основании этих измерений можно вывести заключение об ассоциации молекул эозина в растворах глюкозы.

TABLE DES MATIÈRES

Mathèmatique

1. D. Przeworska-Rolewicz, Remarque sur un résultat de Kropatchev 727
2. E. Marczewski, A General Scheme of the Notions of Independence in Mathematics. 731
3. J. Mikusiński, Sur les notions de distribution et d'opérateur 737
4. P. Erdös and K. Urbanik, On Sets Which Are Measured by Multiples of Irrational Numbers 743
5. S. Świerczkowski, On Independent Elements in Finitely Generated Algebras 749
6. K. Kuratowski, Cohomotopic Multiplication and Duality Theorems Concerning Arbitrary Subsets of Euclidean Space 753

Physique Théorique

7. A. Pawlikowski and W. Szczurówna, On the Canonical Formalism For Dynamical Systems with Constraints 759
8. C. Jankiewicz, Über den Zusammenhang der vom Feld abhängigen Eigenmasse mit dem konform-Minkowskischen Raume 765
9. C. Jankiewicz, Über die Bewegung der Feldsingularitäten in kovarianten Feldtheorien 771
10. A. Deloff and R. Gajewski, Muon Depolarization in High Energy Muon-Electron Scattering 775

Physique Expérimentale

11. D. Frąckowiak, L. Korbutt and T. Korbutt, Fluorescence of Solutions of Eosine in Glucose. 781

Table des matières du Volume VI 785

Liste des auteurs du Volume VI 791

Cena zł 20.—

THE UNIVERSITY OF ILLINOIS AT CHICAGO
3 8198 314 307 966